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Research Article

Existence and Uniqueness of Weak Solutions to Variable-Order Fractional Laplacian Equations with Variable Exponents

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In this paper, the variable-order fractional Laplacian equations with variable exponents and the Kirchhoff-type problem driven by $p(\cdot)$-fractional Laplace with variable exponents were studied. By using variational method, the authors obtain the existence and uniqueness results.

1. Introduction

In recent years, the fractional differential operators and equations have increasingly attracted much attention, since they are good at describing memory and heredity of some complex systems compared with the integer-order derivative [1, 2]. So far, the fractional differential operators have been applied in various research fields, such as optimization [3], fractional quantum mechanics [4], finance [5], image process [6], and biomedical engineering [7]. For more relevant references, we refer the readers to [8–10].

The variable-order fractional derivative extends the study of constant order fractional derivative, which was first proposed by Samko and Ross [11] in 1993. In this concept, the order can change continuously as a function of either dependent or independent variables to better describe the change of memory property with time or space [12]. Later, Lorenzo and Hartley put the variable-order fractional operator to describe the diffusion process in [13], which may also describe the change in temperature [14]. From this, many applications of fractional variable-order spaces have been explored in considerable details [15–17]. The extensive applications urgently need systematic studies on the existence, uniqueness of solutions to these variable-order fractional differential equations. In [18], the infinitely many solutions to Kirchhoff-type variable-order fractional Laplacian equations have been discussed. Xiang [19] has introduced variable-order fractional Laplace $(-\Delta)^{s(\cdot)}$ and explores some problems involving this operator. Moreover, Heydari solved the variable-order fractional nonlinear diffusion-wave equation in [20]. Considering that for some nonhomogeneous materials, the commonly used methods in Lebesgue and Sobolev spaces $L^p(\Omega)$ and $W^{1,p}(\Omega)$ are not sufficient; many scholars have begun to study the differential operator with variable exponent [21–23]. Similar to Lebesgue spaces with variable exponents, Kaufmann [24] introduced the fractional derivative involving variable exponents. In [25], Chen introduced a framework for image restoration using a variable exponents Laplacian. For more literature, see [26–32].

On the other hand, the research on Kirchhoff-type problems has aroused great interest over recent years. Specifically, Kirchhoff built the model given by the equation

$$\rho \frac{\partial^2 v}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial v}{\partial y} \right|^2 dy \right) \frac{\partial v}{\partial y^2} = 0, \tag{1}$$

in [33] to extend the famous D'Alembert wave equation by further investigating the influence of the changes in the length variation during vibrations. Where v is displacement of a string, L is the length of the string, E is the Young modulus of the material, P_0 is the initial tension, ρ is the mass density, and h is the area of cross-section. So far, many researchers have discussed the fractional Kirchhoff-type problems and wide applications. Pucci et al. [34] studied a Kirchhoff-type eigenvalue problem which has a critical nonlinearity and nonlocal fractional Laplace. Later, Molica Bisci et al. [35] centered their work on Kirchhoff nonlocal fractional equations and using three critical point theorem to obtain three solutions. For more results, see [36–38].

To our knowledge, the results of the variable-order fractional Sobolev spaces with variable exponents and fractional $p(\cdot)$-Laplace equations with variable order are few. Motivated by these observations, we focus on the following variable-order fractional Laplacian equation with variable exponents:

$$\begin{cases} (-\Delta)_{p(\cdot)}^{s(\cdot)} v(y) + |v(y)|^{q(y)-2} v(y) = g(y), y \in \Omega\,; \\ v(y) = 0, y \in \partial\Omega, \end{cases} \tag{2}$$

and the Kirchhoff-type problem:

$$\begin{cases} \left(a + b\int_\Omega\int_\Omega \frac{1}{p(y,z)} \frac{|v(y)-v(z)|^{p(y,z)}}{|y-z|^{N+p(y,z)s(y,z)}} dydz\right)(-\Delta)_{p(\cdot)}^{s(\cdot)} v(y) + |v(y)|^{q(y)-2} v(y) = g(y), y \in \Omega\,; \\ v(y) = 0, y \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3}$$

Where the nonlocal operator $(-\Delta)_{p(\cdot)}^{s(\cdot)}$ is defined as

$$(-\Delta)_{p(\cdot)}^{s(\cdot)} v(y) = P.V.\int_\Omega \frac{|v(y)-v(z)|^{p(y,z)-2}(v(y)-v(z))}{|y-z|^{N+s(y,z)p(y,z)}} dz, \tag{4}$$

with

$$\begin{aligned} & s(\cdot) \in C\left(\mathbb{R}^N \times \mathbb{R}^N, (0,1)\right), \\ & p(\cdot) \in C\left(\mathbb{R}^N \times \mathbb{R}^N, (1,\infty)\right), \text{and} \\ & q(y) \in C\left(\mathbb{R}^N, (1,\infty)\right), \end{aligned} \tag{5}$$

and P.V. is a commonly used abbreviation in the principal value sense.

The remainder of this paper is arranged as follows: in Section 2, we review some basic knowledge. In Section 3, we research the existence and uniqueness of the weak solutions to equation (2). In Section 4, we investigate the weak solutions to Kirchhoff-type equation (3).

2. Preliminaries

In this section, we introduce the main tools and some theorems which will be used in this article.

For notational convenience, we define

$$\begin{aligned} & s_- = \min_{(y,z)\in\mathbb{R}^N\times\mathbb{R}^N} s(y,z), s_+ = \max_{(y,z)\in\mathbb{R}^N\times\mathbb{R}^N} s(y,z), \\ & p_- = \min_{(y,z)\in\mathbb{R}^N\times\mathbb{R}^N} p(y,z), p_+ = \max_{(y,z)\in\mathbb{R}^N\times\mathbb{R}^N} p(y,z), \\ & q_- = \min_{y\in\bar{\Omega}} q(y), q_+ = \max_{y\in\bar{\Omega}} q(y). \end{aligned} \tag{6}$$

Concerning the function $s(\cdot)$, $p(\cdot)$, and $q(\cdot)$ satisfied the followings:

(A.1) $s(\cdot)$ is symmetric, i.e., $s(y,z) = s(z,y)$ and continuous for all $(y,z) \in \mathbb{R}^N \times \mathbb{R}^N$ with $0 < s_- \le s_+ < 1$.

(A.2) $p(\cdot)$ is symmetric, i.e., $p(y,z) = p(z,y)$ and continuous for all $(y,z) \in \mathbb{R}^N \times \mathbb{R}^N$ and $ps < N$ for all $(y,z) \in \bar{\Omega} \times \bar{\Omega}$.

(A.3) $p(\cdot)$ and $q(\cdot)$ are both bounded away from 1 and ∞, i.e., there exist $1 < q_- < q_+ < +\infty$ and $1 < p_- < p_+ < +\infty$ so that $q_- \le q(y) \le q_+$ and $p_- \le p(y,z) \le p_+$ for every $(y,z) \in \mathbb{R}^N \times \mathbb{R}^N$.

The Banach space $L^{q(y)}(\Omega)$ is given by

$$L^{q(y)}(\Omega) = \left\{ v : \text{function} v : \Omega \longrightarrow \mathbb{R} \text{ismeasurableand} \exists \xi > 0 : \int_\Omega \left|\frac{v(y)}{\xi}\right|^{q(y)} dy < \infty \right\}, \tag{7}$$

with the norm

$$\|v\|_{L^{q(y)}(\Omega)} = \inf\left\{ \xi > 0 : \int_\Omega \left|\frac{v(y)}{\xi}\right|^{q(y)} dy < 1 \right\}. \tag{8}$$

It follows from [39] that $(L^{q(y)}(\Omega), \|\bullet\|_{L^{q(y)}(\Omega)})$ is a separable and reflexive Banach space.

Consider the space

$$C_+(\bar{\Omega}) = \left\{ \psi \in C(\bar{\Omega}) : 1 < \psi \text{ for all } y \in \bar{\Omega} \right\}. \tag{9}$$

Let $q' \in C_+(\bar{\Omega})$ be the conjugate exponent of q, i.e.,

$$\frac{1}{q(y)} + \frac{1}{q'(y)} = 1, \text{for all } y \in \bar{\Omega}. \quad (10)$$

Then, we have

Theorem 1 (Holder's inequality, [39]). *Suppose that* $u \in L^{q(\cdot)}(\Omega)$ *and* $v \in L^{q'(\cdot)}(\Omega)$. *Then,*

$$\left| \int_\Omega uvdx \right| \le \left(\frac{1}{q_-} + \frac{1}{q'_-} \right) \|u\|_{L^{q(\cdot)}(\Omega)} \|v\|_{L^{q'(\cdot)}(\Omega)} \le 2 \|u\|_{L^{q(\cdot)}(\Omega)} \|v\|_{L^{q'(\cdot)}(\Omega)}. \quad (11)$$

In addition, the Sobolev fractional space with variable exponents and variable order for $0 < s(\cdot) < 1$ is defined as follows:

$$W = W^{s(y,z),q(y),p(y,z)}(\Omega) = \left\{ v : \Omega \longrightarrow \mathbb{R} : v \in L^{q(y)}(\Omega) : \int_\Omega \int_\Omega \frac{|v(y) - v(z)|^{p(y,z)}}{\xi^{p(y,z)} |y - z|^{N+s(y,z)p(y,z)}} \cdot dydz < \infty, \text{for some } \xi > 0 \right\}, \quad (12)$$

and the variable exponent seminorm:

$$[v]^{s(y,z),p(y,z)}(\Omega) = \inf \left\{ \xi > 0 : \int_\Omega \int_\Omega \frac{|v(y) - v(z)|^{p(y,z)}}{\xi^{p(y,z)} |y - z|^{N+s(y,z)p(y,z)}} < 1 \right\}. \quad (13)$$

If we equip W with

$$\|v\|_W = \|v\|_{L^{q(y)}(\Omega)} + [v]^{s(y,z),p(y,z)}(\Omega), \quad (14)$$

then W is a Banach space.

Let $E = \mathbb{R}^N \times \mathbb{R}^N \setminus \Omega^c \times \Omega^c$. It is proved that the norm $\|\bullet\|_W$ is different from $\|\bullet\|_{s(\cdot),p(\cdot)}$ in [40] since $\Omega \times \Omega \subset E$ but $\Omega \times \Omega \ne E$. Then, let

$$W_0 = \left\{ v \in W : v = 0 \text{a.e.in } \mathbb{R}^N \setminus \Omega \right\} \quad (15)$$

be the workspace in Sections 3 and 4. W_0 is a Banach space which is separable and reflexive with the norm

$$\|v\|_{W_0} = \inf \xi > 0 : \iint_E \frac{|v(y) - v(z)|^{p(y,z)}}{\xi^{p(y,z)} |y - z|^{N+p(y,z)s(y,z)}} dydz = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|v(y) - v(z)|^{p(y,z)}}{\xi^{p(y,z)} |y - z|^{N+p(y,z)s(y,z)}} dydz < 1 \Bigg\}, \quad (16)$$

where the inequality is a result of $v = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$. Similar to the proof of the Theorem 2.4 in [40], there is a constant $C > 0$ so that

$$\|v\|_{\bar{P}(\cdot)} \le C \|v\|_{W_0}, \quad (17)$$

for all $v \in W_0$, where $\bar{p}(y) = p(y, y)$. Then, $\|\bullet\|_W$ and $\|\bullet\|_{W_0}$ are topological equivalent in W_0.

Theorem 2 (Sobolev-type embedding theorem [41]). *Suppose* $\Omega \subset \mathbb{R}^N$ *($N \ge 2$) is a smooth bounded domain and* $p(\cdot)$, $s(\cdot)$ *satisfy the conditions (A1) and (A2), respectively. Assume* $\alpha \in C_+(\bar{\Omega})$, $\beta \in C_+(\bar{\Omega})$ *such that* $\alpha(y) \ge p(y, y)$ *and* $\beta(y) < p_s^*(y) := Np(y, y)/N - s(y, y)p(y, y)$ *for all* $y \in \bar{\Omega}$. *Then, there exists a constant* $J = J(N, s, p, \alpha, \beta, \Omega) > 0$ *such that for every* $v \in W$

$$\|v\|_{L^{\beta(y)}(\Omega)} \le J \|v\|_W. \quad (18)$$

Furthermore, the embedding is compact.

Remark 3. According to Theorem 2.2 in [42], Theorem 2 holds true in W_0.

For classical Sobolev space theory such as constant order and constant exponential, see [43–45]. And for variable order, variable exponent cases, see [21, 41].

3. Equations in Variable-Order Fractional Laplacian Equations with Variable Exponents

In this section, we consider the existence and uniqueness of the solution to equation (2). Suppose $g \in L^{k(y)}(\Omega)$ with $1 < k_- < k(y) < k_+ < +\infty$ for each $y \in \bar{\Omega}$, and define the following functional

$$\mathcal{I}(v) = \int_\Omega \int_\Omega \frac{|v(y) - v(z)|^{p(y,z)}}{|y - z|^{N+s(y,z)p(y,z)} p(y, z)} dydz + \int_\Omega \frac{|v(y)|^{q(y)}}{q(y)} dy - \int_\Omega g(y)v(y)dy, \quad (19)$$

and the weak solution to equation (2) in space W_0.

Definition 4. We say that v is a weak solution to equation (2) if $v \in W_0$ and

$$\int_\Omega\int_\Omega \frac{|v(y)-v(z)|^{p(y,z)-2}(v(y)-v(z))(w(y)-w(z))}{|y-z|^{N+s(y,z)p(y,z)}}dydz + \int_\Omega |v|^{q(y)-2}v(y)w(y)dy = \int_\Omega g(y)w(y)dy, \tag{20}$$

for any $w \in W_0$.

Now, we show that there is a unique minimizer of $\mathcal{I}$ in W_0 and the minimizer is also a unique weak solution to equation (2).

Theorem 5. *Suppose $s(\cdot) \in (0,1)$ is variable order, and $q(y)$, $p(y,z)$ are continuous variable exponents with $p_- \cdot s(\cdot) > 1$. Let $g \in L^{k(y)}(\Omega)$, with $1 < k_- \le k(y) \le k_+ < +\infty$. If we have*

$$\frac{Np(y,y)}{N-s(y,y)p(y,y)} > \frac{k(y)}{k(y)-1} > 1 \tag{21}$$

for any $y \in \bar{\Omega}$, then there exists a unique minimizer of equation (19) in W_0 which is also a unique weak solution to equation (2).

Proof. We prove the theorem by using the variational method. By simple verification, the functional $\mathcal{I}$ in equation (19) is strictly convex and bounded below (due to the strict convexity of the function $t \longrightarrow t^{p(y,z)}$ for any y and z).

According to Remark 3, W_0 is compactly embedded in $L^{\beta(y)}(\Omega)$ for $\beta(y) < p_s^*(y)$ and especially compactly embedded in $L^{k(y)/k(y)-1}(\Omega)$.

It follows from Theorem 1 that

$$\begin{aligned}\mathcal{I}(v) &= \int_\Omega\int_\Omega \frac{|v(y)-v(z)|^{p(y,z)}}{|y-z|^{N+s(y,z)p(y,z)}p(y,z)}dydz + \int_\Omega \frac{|v(y)|^{q(y)}}{q(y)}dy - \int_\Omega g(y)v(y)dy \\ &\ge \frac{1}{p_+}\|v\|_{W_0}^{p_-} + \int_\Omega \frac{|v(y)|^{q(y)}}{q(y)}dy - \|g\|_{L^{k(y)}(\Omega)}\|v\|_{L^{\frac{k(y)}{k(y)-1}}(\Omega)} \\ &\ge \frac{1}{p_+}\|v\|_{W_0}^{p_-} + \int_\Omega \frac{|v(y)|^{q(y)}}{q(y)}dy - C\|v\|_{W_0}.\end{aligned} \tag{22}$$

Suppose $\|v\|_{W_0} > 1$, it follows that

$$\frac{\mathcal{I}(v)}{\|v\|_{W_0}} \ge \frac{1}{p_+}\|v\|_{W_0}^{p_- -1} - C. \tag{23}$$

Choose a sequence v_j so that $\|v_j\|_{W_0} \longrightarrow \infty$ $(j \longrightarrow \infty)$. Then, we have

$$\mathcal{I}(v_j) \ge \frac{1}{p_+}\|v_j\|_{W_0}^{p_-} - C\|v_j\|_{W_0} \longrightarrow \infty. \tag{24}$$

Thus, $\mathcal{I}$ is coercive. There is a unique minimizer of $\mathcal{I}$.

Finally, let us verify that when v is a minimum of equation (19), it is also a weak solution to equation (2). For $w \in W_0$, we obtain

$$\begin{aligned}0 = \frac{d}{dt}\mathcal{I}(v+tw)\Big|_{t=0} &= \int_\Omega\int_\Omega \frac{d}{dt}\frac{|v(y)-v(z)+t(w(y)-w(z))|^{p(y,z)}}{p(y,z)|y-z|^{N+s(y,z)p(y,z)}} \cdot dydz\Big|_{t=0} + \int_\Omega \frac{d}{dt}\frac{|v(y)+tw(y)|^{q(y)}}{q(y)} \cdot dy\Big|_{t=0} - \int_\Omega \frac{d}{dt}g(y)(v(y)+tw(y))dy\Big|_{t=0} \\ &= \int_\Omega\int_\Omega \frac{|v(y)-v(z)|^{p(y,z)-2}(v(y)-v(z))(w(y)-w(z))}{|y-z|^{N+s(y,z)p(y,z)}} \cdot dydz + \int_\Omega |v(y)|^{q(y)-2}v(y)w(y)dy - \int_\Omega g(y)w(y)dy.\end{aligned} \tag{25}$$

Since v is a minimizer of equation (19), v is a weak solution to equation (2).

Therefore, the proof is completed.

4. The Kirchhoff-Type Problem Driven by a $p(\cdot)$-Fractional Laplace Operator with Variable Order

In this section, we consider the existence and uniqueness of the solution to Kirchhoff-type problem (3) in the following.

Suppose $g \in L^{k(y)}(\Omega)$ with $1 < k_- \le k(y) \le k_+ < +\infty$ for each $y \in \bar{\Omega}$. And discuss the functional associated to equation (3), defined by $\mathcal{J}(v) : W_0 \longrightarrow \mathbb{R}$

$$\begin{aligned}\mathcal{J}(v) &= a\int_\Omega\int_\Omega \frac{1}{p(y,z)}\frac{|v(y)-v(z)|^{p(y,z)}}{|y-z|^{N+p(y,z)s(y,z)}}dydz + \frac{b}{2}\left[\int_\Omega\int_\Omega \frac{1}{p(y,z)}\frac{|v(y)-v(z)|^{p(y,z)}}{|y-z|^{N+p(y,z)s(y,z)}}dydz\right]^2 \\ &\quad + \int_\Omega \frac{|v(y)|^{q(y)}}{q(y)}dy - \int_\Omega g(y)v(y)dy.\end{aligned} \tag{26}$$

Definition 6. v is said to be a weak solution to equation (3) if $v \in W_0$ and

$$\left(a + b\int_\Omega\int_\Omega \frac{|v(y)-v(z)|^{p(y,z)}}{|y-z|^{N+p(y,z)s(y,z)}}dydz\right)\int_\Omega\int_\Omega \cdot \frac{|v(y)-v(z)|^{p(y,z)-2}(v(y)-v(z))(w(y)-w(z))}{|y-z|^{N+s(y,z)p(y,z)}} \cdot dydz + \int_\Omega |v|^{q(y)-2}v(y)w(y)dy = \int_\Omega g(y)w(y)dy, \tag{27}$$

for any $w \in W_0$.

Theorem 7. *Suppose $a, b > 0$. Assume that $s(\cdot) \in (0,1)$ is variable order, and $q(y)$, $p(\cdot)$ are continuous variable exponents with $p\_s(\cdot) > 1$. Let $g \in L^{k(y)}(\Omega)$, with $1 < k_- \le k(y) \le k_+ < +\infty$ for any $y \in \bar{\Omega}$ so that*

$$\frac{Np(y,y)}{N - s(y,y)p(y,y)} > \frac{k(y)}{k(y)-1} > 1, \tag{28}$$

then there is a unique minimizer of equation (26) in W_0 which is also a unique weak solution to equation (3).

Proof. Let us prove that $\mathcal{J}$ is coercive. We have

$$\begin{aligned}\mathcal{J}(v) &= a\int_\Omega\int_\Omega \frac{1}{p(y,z)}\frac{|v(y)-v(z)|^{p(y,z)}}{|y-z|^{N+p(y,z)s(y,z)}}dydz \\ &\quad + \frac{b}{2}\left[\int_\Omega\int_\Omega \frac{1}{p(y,z)}\frac{|v(y)-v(z)|^{p(y,z)}}{|y-z|^{N+p(y,z)s(y,z)}}dydz\right]^2 \\ &\quad + \int_\Omega \frac{|v(y)|^{q(y)}}{q(y)}dy - \int_\Omega g(y)v(y)dy \\ &\ge \frac{a}{p_+}\|v\|_{W_0}^{p_-} + \frac{b}{2(p_+)^2}\|v\|_{W_0}^{2p_-} + \int_\Omega \frac{|v(y)|^{q(y)}}{q(y)}dy \\ &\quad - \int_\Omega g(y)v(y)dy \ge \frac{a}{p_+}\|v\|_{W_0}^{p_-} + \frac{b}{2(p_+)^2}\|v\|_{W_0}^{2p_-} \\ &\quad + \int_\Omega \frac{|v(y)|^{q(y)}}{q(y)}dy - \|g\|_{L^{k(y)}(\Omega)}\|v\|_{L^{\frac{k(y)}{k(y)-1}}(\Omega)} \\ &\ge \frac{a}{p_+}\|v\|_{W_0}^{p_-} + \frac{b}{2(p_+)^2}\|v\|_{W_0}^{2p_-} + \int_\Omega \frac{|v(y)|^{q(y)}}{q(y)}dy \\ &\quad - C\|v\|_{W_0}.\end{aligned} \tag{29}$$

Suppose $\|v\|_{W_0} > 1$, we have

$$\begin{aligned}\frac{\mathcal{J}(v)}{\|v\|_{W_0}} &\ge \frac{a}{p_+}\|v\|_{W_0}^{p_- -1} + \frac{b}{2(p_+)^2}\|v\|_{W_0}^{2p_- -1} \\ &\quad + \frac{1}{\|v\|_{W_0}}\int_\Omega \frac{|v(y)|^{q(y)}}{q(y)}dy - C \\ &\ge \frac{a}{p_+}\|v\|_{W_0}^{p_- -1} + \frac{b}{2(p_+)^2}\|v\|_{W_0}^{2p_- -1} - C.\end{aligned} \tag{30}$$

Select a sequence v_j so that $\|v\|_{W_0} \longrightarrow \infty$ $(j \longrightarrow \infty)$. Then, we have

$$\mathcal{J}(v_j) \ge \frac{a}{p_+}\|v\|_{W_0}^{p_-} + \frac{b}{2(p_+)^2}\|v_j\|_{W_0}^{2p_-} - C\|v_j\|_{W_0} \longrightarrow \infty. \tag{31}$$

Thus, $\mathcal{J}$ is coercive. The functional $\mathcal{J}$ has a unique minimizer.

Next, we verify that when v is a minimum of equation (26), it is a weak solution to equation (3). For $v \in W_0$, we have

$$\begin{aligned}0 &= \frac{d}{dt}\mathcal{J}(v+tw)\Big|_{t=0} \\ &= \frac{d}{dt}\Bigg(a\int_\Omega\int_\Omega \frac{1}{p(y,z)}\frac{|v(y)-v(z)+t(w(y)-w(z))|^{p(y,z)}}{|y-z|^{N+p(y,z)s(y,z)}} \\ &\quad \cdot dydz + \frac{b}{2}\left[\int_\Omega\int_\Omega \frac{1}{p(y,z)}\frac{|v(y)-v(z)+t(w(y)-w(z))|^{p(y,z)}}{|y-z|^{N+p(y,z)s(y,z)}}dydz\right]^2 \\ &\quad + \int_\Omega \frac{|v(y)+tw(y)|^{q(y)}}{q(y)}dy - \int_\Omega g(y)(v(y)+tw(y))dy\Bigg)\Big|_{t=0} \\ &= \left(a + b\int_\Omega\int_\Omega \frac{|v(y)-v(z)|^{p(y,z)}}{|y-z|^{N+p(y,z)s(y,z)}}dydz\right) \\ &\quad \cdot \int_\Omega\int_\Omega \frac{|v(y)-v(z)|^{p(y,z)-2}(v(y)-v(z))(w(y)-w(z))}{|y-z|^{N+s(y,z)p(y,z)}} \\ &\quad \cdot dydz + \int_\Omega |v|^{q(y)-2}v(y)w(y)dy - \int_\Omega g(y)w(y)dy.\end{aligned} \tag{32}$$

Hence, v is a weak solution to the equation (3).

Remark 8. When $a = 0, b > 0$, the equation (3) becomes a degenerate Kirchhoff-type equation and Theorem 7 still holds.

Data Availability

No data were generated or used during the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly to the writing of this paper. All authors read and approved the final manuscript.

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