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Research Article

Extended Odd Fréchet-G Family of Distributions

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The need to develop generalizations of existing statistical distributions to make them more flexible in modeling real data sets is vital in parametric statistical modeling and inference. Thus, this study develops a new class of distributions called the extended odd Fréchet family of distributions for modifying existing standard distributions. Two special models named the extended odd Fréchet Nadarajah-Haghighi and extended odd Fréchet Weibull distributions are proposed using the developed family. The densities and the hazard rate functions of the two special distributions exhibit different kinds of monotonic and nonmonotonic shapes. The maximum likelihood method is used to develop estimators for the parameters of the new class of distributions. The application of the special distributions is illustrated by means of a real data set. The results revealed that the special distributions developed from the new family can provide reasonable parametric fit to the given data set compared to other existing distributions.

1. Introduction

The fundamental reason for parametric statistical modeling is to identify the most appropriate model that adequately describes a data set obtained from experiment, observational studies, surveys, and so on. Most of these modeling techniques are based on finding the most suitable probability distribution that explains the underlying structure of the given data set. However, there is no single probability distribution that is suitable for different data sets. Thus, this has triggered the need to extend the existing classical distributions or develop new ones. Barrage of methods for defining new families of distributions have been proposed in literature for extending or generalizing the existing classical distributions in recent time. Some of these methods include Weibull-G [1], odd generalized exponential family [2], odd Lindley-G family [3], Topp-Leone odd log-logistic-G family [4], odd Burr-G family [5], odd Fréchet-G family [6], odd gamma-G family [7], transformed-transformer method [8], exponentiated transformed-transformer method [9], exponentiated generalized transformed-transformer method [10], alpha power transformed family [11], alpha logarithmic transformed family [12], Kumaraswamy-G family [13], beta-G family [14], Kumaraswamy transmuted-G family [15], transmuted geometric-G family [16], and beta extended Weibull family [17]. These methods are developed with the motivation of defining new models with different kinds of failure rates (monotonic and nonmonotonic), constructing heavy-tailed distributions for modeling different kinds of data sets, developing distributions with symmetric, right skewed, left skewed, reversed J shape, and consistently providing a reasonable parametric fit to given data sets.

Recently, [6] developed the odd Fréchet family of distributions and defined its cumulative distribution function (CDF) as

$$H(x) = e^{-[(1-G(x;\psi))/G(x;\psi)]^{\theta}}, \quad x \in \mathbb{R}, \tag{1}$$

where $G(x;\psi)$ is the baseline CDF and ψ is a $p \times 1$ vector of associated parameters. Using the transformed-transformer method proposed by [8], an extension of the odd Fréchet family of distributions called the extended odd Fréchet-G (EOF-G) family of distributions is developed by integrating the Fréchet probability density function (PDF). Hence, the CDF of the EOF-G family is defined as

$$\begin{aligned} F(x) &= \int_0^{G(x;\psi)^{\alpha}/(1-G(x;\psi)^{\alpha})} \theta x^{-\theta-1} e^{-x^{-\theta}} dx \\ &= e^{-[(1-G(x;\psi)^{\alpha})/G(x;\psi)^{\alpha}]^{\theta}}, \quad \alpha > 0,\ \theta > 0,\ x \in \mathbb{R}, \end{aligned} \tag{2}$$

where α and θ are extra shape parameters. The corresponding PDF of the new family is obtained by differentiating equation (2) and is given by

$$f(x) = \frac{\alpha\theta g(x;\psi)\left(1-G(x;\psi)^{\alpha}\right)^{\theta-1}}{G(x;\psi)^{\alpha\theta+1}} e^{-[(1-G(x;\psi)^{\alpha})/G(x;\psi)^{\alpha}]^{\theta}}, \quad \alpha>0,\ \theta>0,\ x\in\mathbb{R}. \tag{3}$$

The associated hazard rate function of the EOF-G family is defined as

$$h(x) = \frac{\alpha\theta g(x;\psi)\left(1-G(x;\psi)^{\alpha}\right)^{\theta-1}}{G(x;\psi)^{\alpha\theta+1}\left(1-e^{-[(1-G(x;\psi)^{\alpha})/G(x;\psi)^{\alpha}]^{\theta}}\right)} \cdot e^{-[(1-G(x;\psi)^{\alpha})/G(x;\psi)^{\alpha}]^{\theta}}, \quad \alpha>0,\ \theta>0,\ x\in\mathbb{R}. \tag{4}$$

Hereafter, a random variable X following the EOF-G distribution is denoted by $X \sim \text{EOF}-\text{G}(x;\alpha,\theta,\psi)$ and for the purpose of simplicity, $G(x;\psi)$ can be written as $G(x)$. The CDF of the EOF-G family of distributions is tractable which makes it easy to generate random numbers provided that the CDF of the baseline distribution is also tractable. The u^{th} quantile of the EOF-G family is given by

$$x_u = G^{-1}\left[\left(\frac{1}{1+(-\log(u))^{1/\theta}}\right)^{1/\alpha}\right], \quad u\in[0,1], \tag{5}$$

where $G^{-1}(u)$ is the baseline quantile function. When $\alpha = 1$, the EOF-G family of distributions reduces to the odd Fréchet family of distributions. Adopting the interpretation of the CDF of the odd Weibull family as given in [18], the physical interpretation of the CDF of the EOF-G family is given as follows: Suppose Y is a lifetime random variable with continuous CDF, $G(x;\psi)^{\alpha}$. The odds ratio that an individual (component) having the lifetime Y will die (fail) at time x is $G(x;\psi)^{\alpha}/1-G(x;\psi)^{\alpha}$. Given that the variability of these odds of death is denoted by the random variable X and that it follows the Fréchet distribution, then

$$\mathbb{P}(Y\le x) = \mathbb{P}\left(X \le \frac{G(x;\psi)^{\alpha}}{1-G(x;\psi)^{\alpha}}\right) = F(x), \tag{6}$$

which is given in (2). The rest of the paper is organized as follows: In Section 2, special distributions of the EOF-G family are discussed. In Section 3, the mixture representation of the PDF and CDF of the EOF-G family is given. The statistical properties of the new family are derived in Section 4. In Section 5, the estimators for the parameters of the family are developed using the technique of maximum likelihood estimation. Monte Carlo simulations are performed in Section 6 to assess the performance of the estimators. In Section 7, the application of the special distributions is demonstrated using real data set. Finally, the concluding remarks of the study are given in Section 8.

2. Special Distributions of the EOF-G Family

In this section, two special distributions of the EOF-G family are discussed.

2.1. EOF-Nadarajah-Haghighi (EOFNH) Distribution. Suppose the baseline CDF is that of the Nadarajah-Haghighi distribution; that is, $G(x;\beta,\lambda) = 1-e^{(1-(1+\lambda x)^{\beta})}$ with corresponding PDF $g(x;\beta,\lambda) = \beta\lambda(1+\lambda x)^{\beta-1}e^{(1-(1+\lambda x)^{\beta})}$ and positive parameters $\beta,\lambda > 0$. The PDF of the EOFNH distribution is given by

$$f(x) = \frac{\alpha\beta\lambda\theta(1+\lambda x)^{\beta-1}e^{(1-(1+\lambda x)^{\beta})}\left[1-\left(1-e^{(1-(1+\lambda x)^{\beta})}\right)^{\alpha}\right]^{\theta-1}}{\left(1-e^{(1-(1+\lambda x)^{\beta})}\right)^{\alpha\theta+1}} \cdot e^{-[(1-e^{(1-(1+\lambda x)^{\beta})})^{-\alpha}-1]^{\theta}}, \tag{7}$$

where $\alpha,\beta,\theta > 0$ are shape parameters, $\lambda > 0$ is a scale parameter, and $x > 0$. Figure 1 shows the plots of the PDF of the EOFNH distribution for some selected parameter values. The density function exhibits different kinds of shapes.

The corresponding hazard rate function is given by

$$h(x) = \frac{\alpha\beta\lambda\theta(1+\lambda x)^{\beta-1}e^{(1-(1+\lambda x)^{\beta})}\left[1-\left(1-e^{(1-(1+\lambda x)^{\beta})}\right)^{\alpha}\right]^{\theta-1}}{\left(1-e^{(1-(1+\lambda x)^{\beta})}\right)^{\alpha\theta+1}\left(1-e^{-[(1-e^{(1-(1+\lambda x)^{\beta})})^{-\alpha}-1]^{\theta}}\right)} \cdot e^{-[(1-e^{(1-(1+\lambda x)^{\beta})})^{-\alpha}-1]^{\theta}}, \quad x>0. \tag{8}$$

The plots of the hazard rate function of the EOFNH distribution for some selected parameter values are shown in Figure 2. The hazard rate function can assume decreasing, bathtub, upside down bathtub, and other nonmonotonic failure rate forms.

The quantile function of the EOFNH distribution is given by

$$x_u = \frac{\left[1-\log\left(1-\left(1+(-\log(u))^{1/\theta}\right)^{-1/\alpha}\right)\right]^{1/\beta}-1}{\lambda}, \quad u\in[0,1]. \tag{9}$$

Equation (9) can be used to generate random numbers from the EOFNH distribution. The first quartile, median, and upper quartile of the distribution are obtained by substituting $u = 0.25, 0.5$, and 0.75, respectively, into (9).

2.2. EOF-Weibull (EOFW) Distribution. Consider the Weibull distribution with shape parameter $\beta > 0$ and scale parameter $\lambda > 0$, where the CDF and PDF for $x > 0$ are given by $G(x;\beta,\lambda) = 1-e^{-\lambda x^{\beta}}$ and $g(x;\beta,\lambda) = \beta\lambda x^{\beta-1}e^{-\lambda x^{\beta}}$. Substituting the PDF and CDF of the Weibull distribution in (3), the PDF of the EOFW distribution is defined as

$$f(x) = \frac{\alpha\beta\lambda\theta x^{\beta-1}e^{-\lambda x^{\beta}}\left[1-\left(1-e^{-\lambda x^{\beta}}\right)^{\alpha}\right]^{\theta-1}}{\left(1-e^{-\lambda x^{\beta}}\right)^{\alpha\theta+1}} \cdot e^{-[(1-e^{-\lambda x^{\beta}})^{-\alpha}-1]^{\theta}}, \tag{10}$$

where $\alpha,\beta,\theta > 0$ are shape parameters, $\lambda > 0$ is scale parameter, and $x > 0$. Figure 3 displays some of the possible shapes of

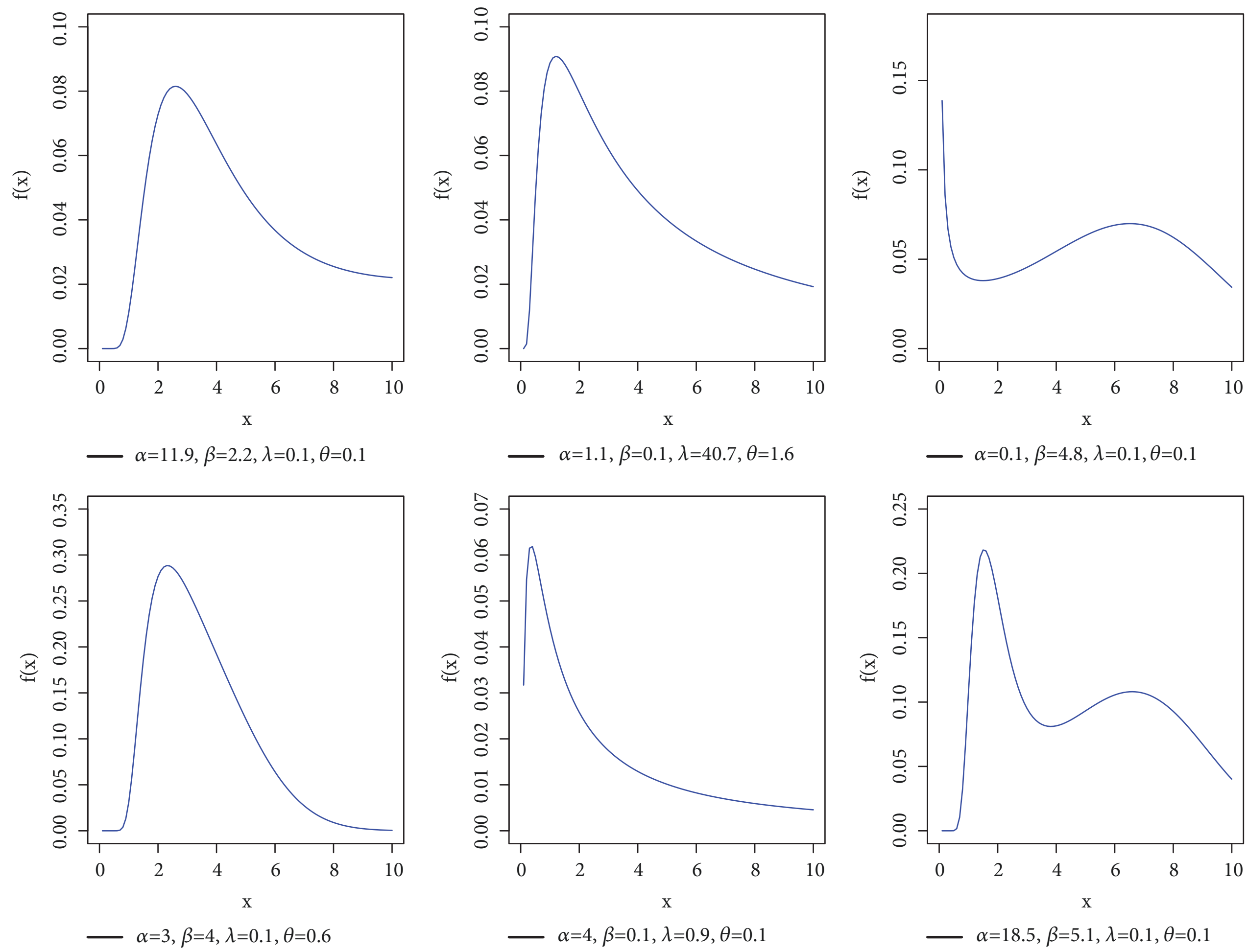

Figure 1: Plots of the EOFNH distribution density function.

the density function of the EOFW distribution. The density exhibits unimodal and reversed J-shape among others.

The hazard rate function of the EOFW distribution is given by

$$h(x) = \frac{\alpha\beta\lambda\theta x^{\beta-1} e^{-\lambda x^{\beta}} \left[1-\left(1-e^{-\lambda x^{\beta}}\right)^{\alpha}\right]^{\theta-1}}{\left(1-e^{-\lambda x^{\beta}}\right)^{\alpha\theta+1}\left(1-e^{-[(1-e^{-\lambda x^{\beta}})^{-\alpha}-1]^{\theta}}\right)} \cdot e^{-[(1-e^{-\lambda x^{\beta}})^{-\alpha}-1]^{\theta}}, \quad x > 0. \tag{11}$$

The hazard rate function can assume decreasing, bathtub, and upside down bathtub forms for some selected parameter values as shown in Figure 4.

The quantile function of the EOFW distribution is defined as

$$x_u = \left\{\frac{-\log\left[1-\left(1+(-\log(u))^{1/\theta}\right)^{-1/\alpha}\right]}{\lambda}\right\}^{1/\beta}, \quad u \in [0,1]. \tag{12}$$

The generation of random numbers from the EOFW distribution can easily be done using (12).

3. Mixture Representation

In this section, the mixture representation of the PDF and CDF of the EOF-G family of distributions is discussed. The mixture representation is useful when deriving the statistical properties of this new family of distributions. Using the Taylor series expansion, the PDF can be written as

$$f(x) = \alpha\theta \sum_{i=0}^{\infty} \frac{(-1)^{i}\, g(x;\psi)\left[1-G(x;\psi)^{\alpha}\right]^{\theta(i+1)-1}}{i!\,G(x;\psi)^{\alpha\theta(i+1)+1}}. \tag{13}$$

Equation (13) can be written as

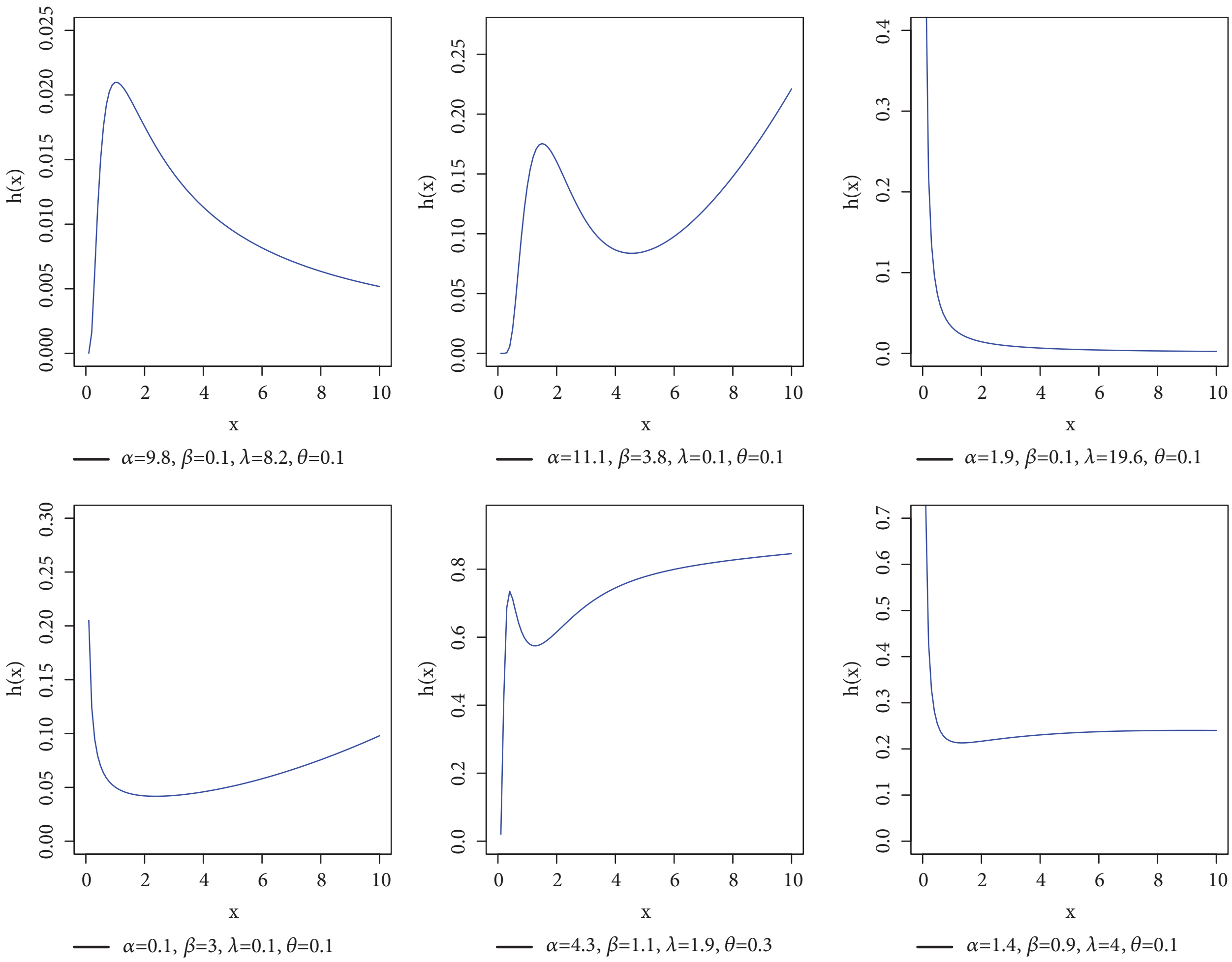

Figure 2: Plots of the EOFNH distribution hazard rate function.

$$f(x) = \alpha\theta \sum_{i=0}^{\infty} \frac{(-1)^i g(x;\psi) \left[1 - G(x;\psi)^{\alpha}\right]^{\theta(i+1)-1} \left[1 - (1 - G(x;\psi))\right]^{-[\alpha\theta(i+1)+1]}}{i!}. \tag{14}$$

Applying the generalized binomial series expansion yields

$$f(x) = \alpha\theta \sum_{i,j=0}^{\infty} \frac{(-1)^i}{i!} \binom{\alpha\theta(i+1)+j}{j} g(x;\psi) \cdot \left[1 - G(x;\psi)\right]^j \left[1 - G(x;\psi)^{\alpha}\right]^{\theta(i+1)-1}. \tag{15}$$

Now using the binomial series expansion, $(1 - z)^{\eta-1} = \sum_{j=0}^{\infty} (-1)^j \binom{\eta-1}{j} z^j, |z| < 1$, thrice yields

$$f(x) = \alpha\theta \sum_{i,j=0}^{\infty} \sum_{k,m=0}^{\infty} \sum_{q=0}^{m+j} \omega_{ijkmq} g(x;\psi) G(x;\psi)^q, \tag{16}$$

where

$$\omega_{ijkmq} = \frac{(-1)^{i+k+m+q}}{i!} \cdot \binom{\alpha\theta(i+1)+j}{j} \binom{\theta(i+1)-1}{k} \binom{\alpha k}{m} \binom{m+j}{q}. \tag{17}$$

Alternatively (16) can be written in terms of the exponentiated-G (exp-G) density function as

$$f(x) = \alpha\theta \sum_{i,j=0}^{\infty} \sum_{k,m=0}^{\infty} \sum_{q=0}^{m+j} \omega^{*}_{ijkmq} \pi_{q+1}(x), \tag{18}$$

where $\omega^{*}_{ijkmq} = \omega_{ijkmq}/(q+1)$ and $\pi_{q+1}(x) = (q+1)g(x;\psi)G(x;\psi)^q$ is the exp-G density function with power

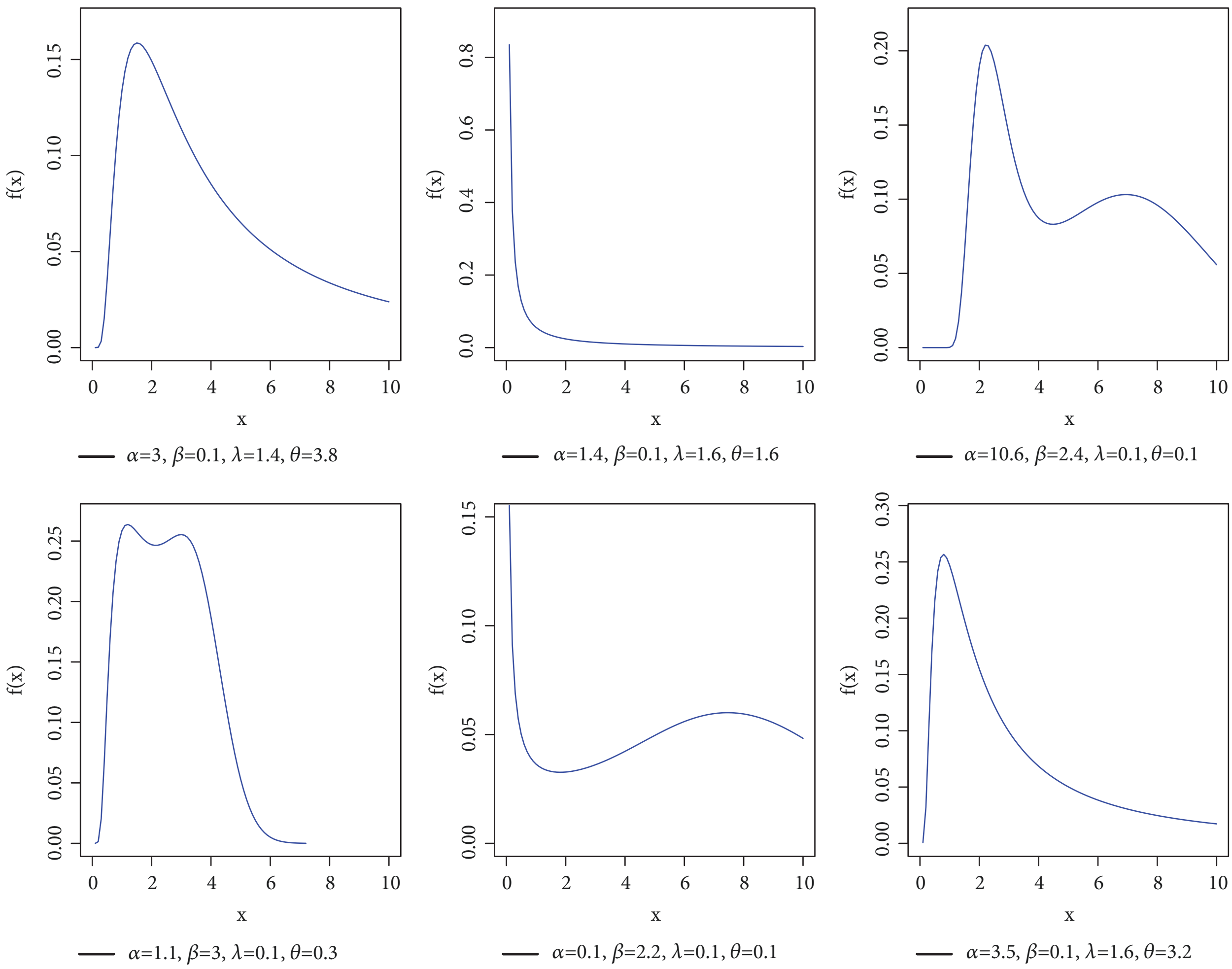

Figure 3: Plots of the EOFW distribution density function.

parameter $q+1$. By integrating (18), the mixture representation of the CDF is given by

$$F(x) = \alpha\theta \sum_{i,j=0}^{\infty} \sum_{k,m=0}^{\infty} \sum_{q=0}^{m+j} \omega^{*}_{ijkmq} \Pi_{q+1}(x), \qquad (19)$$

where $\Pi_{q+1}(x) = G(x;\psi)^{q+1}$ is the CDF of the exp-G family with power parameter $q+1$.

4. Statistical Properties

In this section, the moments, incomplete moments, generating function, entropies, and order statistics of the EOF-G family are derived.

4.1. Moments. The r^{th} noncentral moment of a random variable X is given by $E(X^r) = \int_{-\infty}^{\infty} x^r f(x)dx$. Hence, using this definition the r^{th} noncentral moment of the EOF-G random variable is given by

$$E(X^r) = \alpha\theta \sum_{i,j=0}^{\infty} \sum_{k,m=0}^{\infty} \sum_{q=0}^{m+j} \omega_{ijkmq} \tau_{r,q}, \qquad (20)$$

where $\tau_{r,q} = \int_{-\infty}^{\infty} x^r g(x;\psi) G(x;\psi)^q dx$ is the probability weighted moment of the baseline distribution. The r^{th} noncentral moment can also be expressed in terms of the quantile of the baseline distribution. Letting $G(x;(\psi)) = u$, the r^{th} noncentral moment in terms of the quantile is given by

$$E(X^r) = \alpha\theta \sum_{i,j=0}^{\infty} \sum_{k,m=0}^{\infty} \sum_{q=0}^{m+j} \omega_{ijkmq} \int_0^1 Q_G(u)^r u^q du, \qquad (21)$$

where $Q_G(u)$ is the quantile function of the baseline distribution.

4.2. Incomplete Moments. The r^{th} incomplete moment of a random variable X is defined as $m_r(y) = \int_{-\infty}^{y} x^r f(x)dx$. Thus, the r^{th} incomplete moment of the EOF-G random variable is given by

$$m_r(y) = \alpha\theta \sum_{i,j=0}^{\infty} \sum_{k,m=0}^{\infty} \sum_{q=0}^{m+j} \omega_{ijkmq} \int_0^y x^r g(x;\psi) G(x;\psi)^q dx \qquad (22)$$

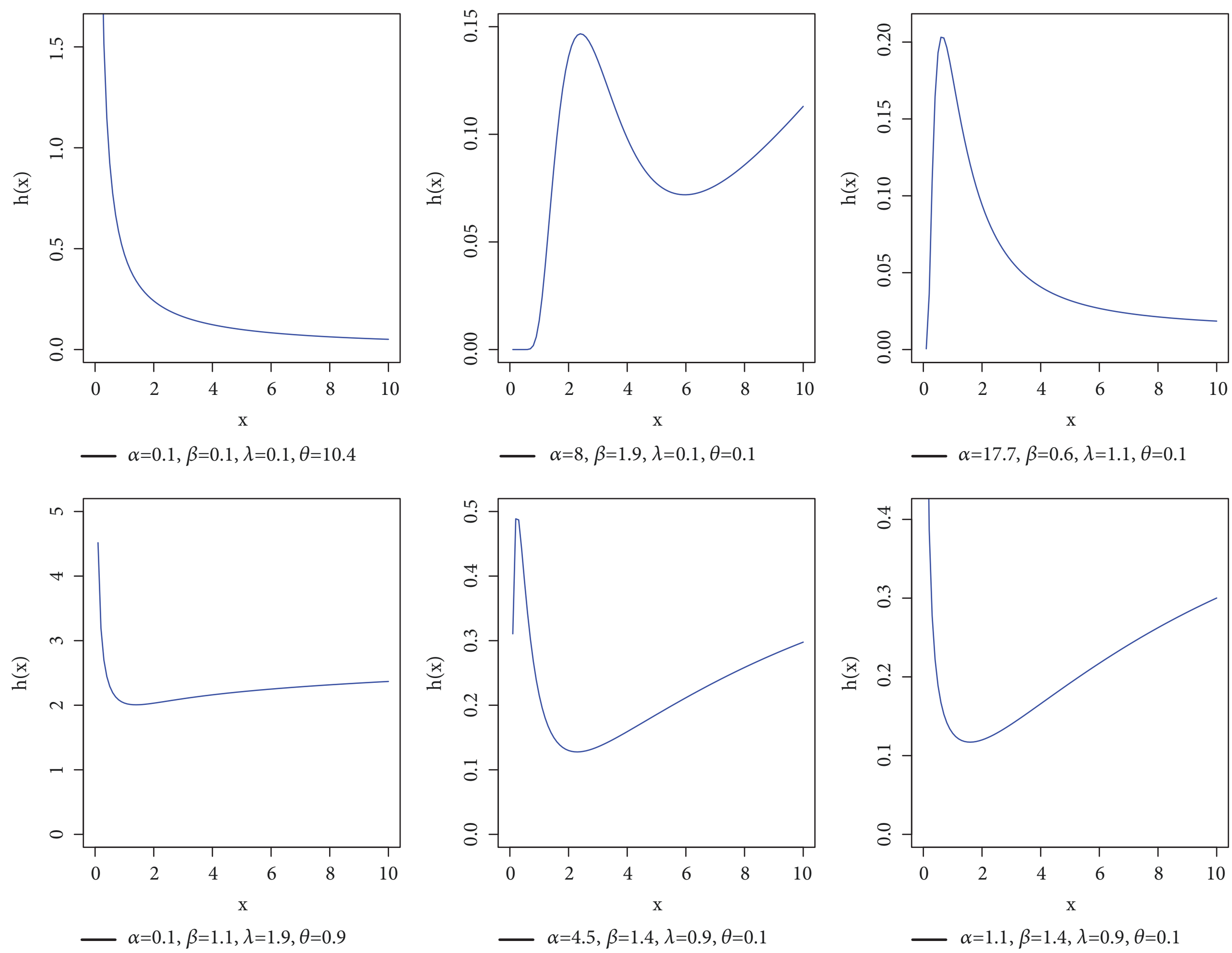

Figure 4: Plots of the EOFW distribution hazard rate function.

In terms of the quantile function of the baseline distribution, the r^{th} incomplete moment is given by

$$m_r(y) = \alpha\theta \sum_{i,j=0}^{\infty} \sum_{k,m=0}^{\infty} \sum_{q=0}^{m+j} \omega_{ijkmq} \int_0^{G(y)} Q_G(u)^r u^q du. \quad (23)$$

Utilize the power series expansion of the quantile of the baseline; that is,

$$Q_G(u) = \sum_{h=0}^{\infty} e_h u^h, \quad (24)$$

where $e_h (h = 0, 1, \ldots)$ are suitably chosen real numbers that depend on the parameters of the $G(x; \psi)$ distribution. Furthermore, for positive integer r $(r \geq 1)$,

$$Q_G(u)^r = \left(\sum_{h=0}^{\infty} e_h u^h \right)^r = \sum_{h=0}^{\infty} e'_{r,h} u^h, \quad (25)$$

where $e'_{r,h} = (he_0)^{-1} \sum_{z=1}^{h} [z(r+1) - h] e_z e'_{r,h-z}$ and $e'_{r,0} = (e_0)^h$. For more details on quantile power series expansion, see [19]. Hence,

$$\begin{aligned} m_r(y) &= \alpha\theta \sum_{i,j=0}^{\infty} \sum_{k,m=0}^{\infty} \sum_{q=0}^{m+j} \omega_{ijkmq} \int_0^{G(y)} \sum_{h=0}^{\infty} e'_{r,h} u^{h+q} du \\ &= \alpha\theta \sum_{i,j=0}^{\infty} \sum_{k,m=0}^{\infty} \sum_{q=0}^{m+j} \omega_{ijkmq} e'_{r,h} \frac{G(y)^{h+q+1}}{h+q+1}. \end{aligned} \quad (26)$$

The incomplete moments are used in the computation of other useful statistical measures such as the mean deviations about the mean ($\delta_1 = E(|X - \mu'_1|)$) and about the median ($\delta_2 = E(|X - M|)$). The mean deviation about the mean and about the median can further be expressed as

$$\begin{aligned} \delta_1 &= 2\mu'_1 F(\mu'_1) - 2m_1(\mu'_1), \\ \delta_2 &= \mu'_1 - 2m_1(M), \end{aligned} \quad (27)$$

where $\mu'_1 = \mu$ is the mean obtained by putting $r = 1$ into (20), M is the median obtained by substituting $u = 0.5$ into (5), and $m_1(y) = \int_{-\infty}^{y} x f(x) dx$ is the first incomplete moment which can be obtained from (23) by substituting $r = 1$.

4.3. Generating Function. In this subsection, two formulae for the computation of the moment generating function $M_X(t) = E(e^{tX})$ are given. Using the Taylor series expansion, $M_X(t) = E(e^{tX}) = \sum_{r=0}^{\infty}(t^r/r!)E(X^r)$. Thus, the moment generating function is given by

$$M_X(t) = \sum_{i,j=0}^{\infty}\sum_{k,m=0}^{\infty}\sum_{r=0}^{\infty}\sum_{q=0}^{m+j}\omega_{ijkmq}\tau_{r,q}. \quad (28)$$

Alternatively, the moment generating function can be expressed in terms of the quantile function of the baseline distribution as

$$M_X(t) = \sum_{i,j=0}^{\infty}\sum_{k,m=0}^{\infty}\sum_{q=0}^{m+j}\omega_{ijkmq}\int_0^1 e^{tQ_G(u)}u^q du. \quad (29)$$

4.4. Entropy Measures. Entropies are measures of uncertainty or variation of a random variable. In this subsection, the Rényi, Shannon, and δ entropies are studied. The Rényi entropy [20] of a random variable X with PDF $f(x)$ is defined as

$$I_R(\delta) = \frac{1}{1-\delta}\log\left[\int_{-\infty}^{\infty} f(x)^{\delta}dx\right], \quad \delta > 0,\ \delta \neq 1. \quad (30)$$

Using similar concepts for expanding the PDF,

$$f(x)^{\delta} = (\alpha\theta)^{\delta}\sum_{i,j=0}^{\infty}\sum_{k,m=0}^{\infty}\sum_{q=0}^{m+j}\varpi_{ijkmq}g(x;\psi)^{\delta}G(x;\psi)^q, \quad (31)$$

where

$$\varpi_{ijkmq} = \frac{(-1)^{i+k+m+q}\delta^i}{i!} \cdot \binom{\alpha\theta(i+\delta)+\delta+j-1}{j}\binom{\theta(i+\delta)-\delta}{k}\binom{\alpha k}{m}\binom{m+j}{q}. \quad (32)$$

Hence,

$$I_R(\delta) = \frac{1}{1-\delta}\log\left[(\alpha\theta)^{\delta}\sum_{i,j=0}^{\infty}\sum_{k,m=0}^{\infty}\sum_{q=0}^{m+j}\varpi_{ijkmq} \cdot \int_{-\infty}^{\infty} g(x;\psi)^{\delta}G(x;\psi)^q dx\right], \quad \delta > 0,\ \delta \neq 1. \quad (33)$$

The Shannon entropy [21] of a random variable X, say $\eta_X = E(-\log f(X))$. The Shannon entropy is a special case of the Rényi entropy when $\delta \uparrow 1$. The $\delta-$entropy is given by

$$H(\delta) = \frac{1}{\delta-1}\log\left[1-\int_{-\infty}^{\infty} f(x)^{\delta}dx\right], \quad \delta > 0,\ \delta \neq 1. \quad (34)$$

Thus, the $\delta-$entropy is

$$H(\delta) = \frac{1}{\delta-1}\left[1-(\alpha\theta)^{\delta}\sum_{i,j=0}^{\infty}\sum_{k,m=0}^{\infty}\sum_{q=0}^{m+j}\varpi_{ijkmq} \cdot \int_{-\infty}^{\infty} g(x;\psi)^{\delta}G(x;\psi)^q dx\right], \quad \delta > 0,\ \delta \neq 1. \quad (35)$$

4.5. Order Statistics. Let $X_1, X_2, \ldots, X_n$ represent a random sample from EOF-G family and $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the order statistics. Then the PDF, $f_{p:n}(x)$, of the p^{th} order statistic $X_{p:n}$ is

$$f_{p:n}(x) = \frac{n!}{(p-1)!(n-p)!}\sum_{i=0}^{n-p}(-1)^i F(x)^{p+i-1}f(x). \quad (36)$$

Substituting the PDF and the CDF of the EOF-G random variable into the last equation yields

$$f_{p:n}(x) = \frac{n!\alpha\theta}{(p-1)!(n-p)!} \cdot \sum_{j,k=0}^{\infty}\sum_{q,s=0}^{\infty}\sum_{w=0}^{k+s}\sum_{i=0}^{n-p}\varphi_{ijkqsw}g(x;\psi)G(x;\psi)^w, \quad (37)$$

after some algebraic manipulation, where

$$\varphi_{ijkqsw} = \frac{(-1)^{i+j+q+s+w}(p+i)^j}{j!} \cdot \binom{n-p}{i}\binom{\alpha\theta(j+1)+k}{k}\binom{\theta(j+1)-1}{q}\binom{\alpha q}{s}\binom{k+s}{w}. \quad (38)$$

The PDF of the p^{th} order statistic can be expressed in terms of the exp-G density function as

$$f_{n:p}(x) = \frac{n!\alpha\theta}{(p-1)!(n-p)!}\sum_{j,k=0}^{\infty}\sum_{q,s=0}^{\infty}\sum_{w=0}^{k+s}\sum_{i=0}^{n-p}\varphi^*_{ijkqsw}\Delta_{w+1}(x), \quad (39)$$

where $\varphi^*_{ijkqsw} = \varphi_{ijkqsw}/(w+1)$ and $\Delta_{w+1}(x) = (w+1)g(x;\psi)G(x;\psi)^w$ is the exp-G density function with power parameter $w+1$.

5. Parameter Estimation

In this section, the maximum likelihood technique is employed to develop estimators for estimating the parameters of the EOF-G family of distributions. Suppose $x_1, x_2, \ldots, x_n$ are possible outcomes of a random sample obtained from

$X \sim \text{EOF} - \text{G}(x;\alpha,\theta,\psi)$ and $\vartheta = (\alpha,\theta,\psi)^T$ is a parameter vector; then the total log-likelihood function is given by

$$\begin{aligned}\ell &= n\log(\alpha\theta) + \sum_{i=1}^{n}\log g(x_i;\psi) \\ &+ (\theta-1)\sum_{i=1}^{n}\log\left[1-G(x_i;\psi)^{\alpha}\right] \\ &- (\alpha\theta+1)\sum_{i=1}^{n}\log G(x_i;\psi) \\ &- \sum_{i=1}^{n}\left[\frac{1-G(x_i;\psi)^{\alpha}}{G(x_i;\psi)^{\alpha}}\right]^{\theta}.\end{aligned} \tag{40}$$

By finding the partial derivatives of (40), the components of the score vector $U(\vartheta) = (\partial\ell/\partial\alpha, \partial\ell/\partial\theta, \partial\ell/\partial\psi)^T$ are

$$\begin{aligned}\frac{\partial\ell}{\partial\alpha} &= \frac{n}{\alpha} + (\theta-1)\sum_{i=1}^{n}\frac{G(x_i;\psi)^{\alpha}\log G(x_i;\psi)}{1-G(x_i;\psi)} \\ &- \theta\sum_{i=1}^{n}\log G(x_i;\psi) \\ &+ \theta\sum_{i=1}^{n}\frac{\left[1-G(x_i;\psi)^{\alpha}\right]^{\theta-1}\log G(x_i;\psi)}{G(x_i;\psi)^{\alpha\theta}},\end{aligned} \tag{41}$$

$$\begin{aligned}\frac{\partial\ell}{\partial\theta} &= \frac{n}{\theta} + \sum_{i=1}^{n}\log\left[1-G(x_i;\psi)^{\alpha}\right] - \alpha\sum_{i=1}^{n}\log G(x_i;\psi) \\ &- \sum_{i=1}^{n}\left[\frac{1-G(x_i;\psi)^{\alpha}}{G(x_i;\psi)^{\alpha}}\right]^{\theta}\log\left[\frac{1-G(x_i;\psi)^{\alpha}}{G(x_i;\psi)^{\alpha}}\right],\end{aligned} \tag{42}$$

$$\begin{aligned}\frac{\partial\ell}{\partial\psi} &= \sum_{i=1}^{n}\frac{g'(x_i;\psi)}{g(x_i;\psi)} \\ &+ \alpha(\theta-1)\sum_{i=1}^{n}\frac{G'(x_i;\psi)G(x_i;\psi)^{\alpha-1}}{1-G(x_i;\psi)} \\ &- (\alpha\theta+1)\sum_{i=1}^{n}\frac{G'(x_i;\psi)}{G(x_i;\psi)} \\ &+ \alpha\theta\sum_{i=1}^{n}\frac{G'(x_i;\psi)\left[1-G(x_i;\psi)^{\alpha}\right]^{\theta-1}}{G(x_i;\psi)^{\alpha\theta+1}},\end{aligned} \tag{43}$$

where $g'(x_i;\psi) = \partial g(x_i;\psi)/\partial\psi$ and $G'(x_i;\psi) = \partial G(x_i;\psi)/\partial\psi$. In order to obtain the estimators for the parameters, we set (41), (42), and (43) to zero and solve the system numerically using methods such as the quasi-Newton algorithms since the equations do not have closed form. To obtain interval estimates of the parameters, a $p \times p$ observed information matrix can be estimated as $J(\vartheta) = \partial^2\ell/\partial q\partial r$ (for $q, r = \alpha,\theta,\psi$), whose elements are evaluated numerically. To compute the approximate confidence intervals of the parameters, the multivariate normal distribution $N_p(\mathbf{0}, J(\widehat{\vartheta})^{-1})$. Here, $J(\widehat{\vartheta})$ is the observed information evaluated at $\widehat{\vartheta}$. To investigate whether the EOF-G distributions are superior to the odd Fréchet family of distributions for given data sets, the likelihood ratio (LR) test can be performed using the following hypotheses: $H_0 : \alpha = 1$ versus $H_a : H_0$ is false. The LR test statistic is given by $LR = 2\{\ell(\widehat{\vartheta}) - \ell(\overline{\vartheta})\}$, where $\widehat{\vartheta}$ is the vector of unrestricted estimates under H_a and $\overline{\vartheta}$ is the vector of restricted maximum likelihood estimates under H_0. The LR test statistic is asymptotically distributed as Chi-square random variable with degrees of freedom equal to the difference between the numbers of parameters of the two models. As a decision rule, the null hypothesis is rejected when the LR test statistic exceeds the upper $100(1-\eta)\%$ quantile of the Chi-square distribution.

6. Simulation Study

In this section, Monte Carlo simulations are performed to assess the accuracy and consistency of the maximum likelihood estimators. For the purpose of illustration, the simulations are performed using the estimators of the parameters of the EOFNH distribution. The quantile function given in (9) is used to generate random observations from the EOFNH distribution. The simulations are repeated $N = 1{,}000$ times each with sample size $n = 25, 75, 150, 300, 600, 800$ and parameter values I : $\alpha = 0.5, \beta = 0.5, \lambda = 0.5, \theta = 0.5$, II : $\alpha = 3.3, \beta = 0.8, \lambda = 0.2, \theta = 0.8$, and III : $\alpha = 0.9, \beta = 0.4, \lambda = 0.2, \theta = 0.6$. Table 1 presents the average bias (AB), the root mean square error (RMSE), and coverage probability (CP) of the 95% confidence intervals for the estimators of the parameters. The results indicated that the ABs and RMSEs decrease as the sample size increases. These results clearly show the accuracy and the consistency of the maximum likelihood estimators. Also, the CPs are quite close to the nominal value. Thus, the maximum likelihood technique works very well to estimate the parameters of the EOFNH distribution.

7. Application

In this section, the application of the EOFNH and EOFW distributions is illustrated using a real data set. The data consists of the Fatigue time of 101 6061-T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second. The data set given in Table 2 can be found in Birnbaum and Saunders [22]. The performance of the EOFNH and EOFW distributions is compared with that of the odd Fréchet Nadarajah-Haghighi (OFNH) and odd Fréchet Weibull (OFW) distributions using the Akaike information criterion (AIC) [23, 24] and Bayesian information criterion (BIC) [25]. The maximum likelihood estimates of the parameters of the fitted distributions are computed by maximizing the log-likelihood function via the subroutine *mle2* uisng the *bbmle* package in the R software [26].

Table 1: Monte Carlo simulation results: AB and RMSE and CP.

Parameter	n	I			II			III		
		AB	RMSE	CP	AB	RMSE	CP	AB	RMSE	CP
α	25	0.5406	1.9394	0.9780	24.9701	133.9291	0.9160	3.1057	17.0755	0.8780
	75	0.1710	0.5134	0.9470	12.5503	76.6534	0.9030	1.1174	3.9980	0.8850
	150	0.0919	0.3459	0.9120	5.6552	41.2149	0.9200	0.4664	1.3878	0.9020
	300	0.0242	0.2526	0.8670	2.1667	5.1107	0.9540	0.2038	0.6941	0.9280
	600	-0.0043	0.2076	0.8300	0.9715	2.8387	0.9640	0.1071	0.4109	0.9430
	800	-0.0010	0.1794	0.8670	0.7123	2.3503	0.9560	0.0623	0.3374	0.9430
β	25	0.0040	0.2147	0.9620	0.4450	1.3048	0.9810	0.0212	0.1894	0.9920
	75	0.0488	0.3523	0.9630	0.2138	0.5569	0.9880	0.0163	0.1195	0.9830
	150	0.0740	0.4896	0.9620	0.1105	0.3330	0.9890	0.0117	0.0705	0.9680
	300	0.0890	0.4095	0.975	0.0633	0.2147	0.9720	0.0068	0.0438	0.9620
	600	0.066	0.2745	0.9670	0.0324	0.1275	0.9630	0.0032	0.0296	0.9520
	800	0.0357	0.1378	0.9730	0.0264	0.1105	0.9620	0.0021	0.0255	0.9510
λ	25	12.5655	249.9064	0.9340	3.7073	32.1823	0.8930	4.9031	47.5403	0.8170
	75	0.6886	2.7878	0.8530	1.2894	15.0985	0.8850	1.4279	12.6625	0.8460
	150	0.3001	0.9320	0.8280	0.1493	2.7383	0.9200	0.2900	1.9911	0.8740
	300	0.1265	0.5992	0.8020	0.0298	0.1599	0.9490	0.0726	0.2735	0.8950
	600	0.0400	0.4286	0.7900	0.0075	0.0931	0.9800	0.0375	0.1450	0.9320
	800	0.0351	0.3629	0.8400	0.0053	0.0856	0.9670	0.0227	0.1175	0.9300
θ	25	0.0856	0.3163	0.9999	0.4125	1.7492	0.8970	0.1846	0.5637	0.9810
	75	0.0616	0.2515	0.9800	0.1192	0.8350	0.8370	0.0725	0.3927	0.9290
	150	0.0599	0.2400	0.9500	0.1242	0.7001	0.8280	0.0599	0.3320	0.9310
	300	0.0790	0.2422	0.9200	0.0602	0.4882	0.8550	0.0435	0.2675	0.9260
	600	0.0824	0.2313	0.8840	0.0506	0.3537	0.8940	0.1230	0.1618	0.9400
	800	0.0601	0.1957	0.9040	0.0396	0.3043	0.9130	0.0179	0.1502	0.9410

Table 2: Fatigue time of 101 6061-T6 aluminum coupons.

70	90	96	97	99	100	103	104	104	105	107	108	108	108	109
109	112	112	113	114	114	114	116	119	120	120	120	121	121	123
124	124	124	124	124	128	128	129	129	130	130	130	131	131	131
131	131	132	132	132	133	134	134	134	134	134	136	136	137	138
138	138	139	139	141	141	142	142	142	142	142	142	144	144	145
146	148	148	149	151	151	152	155	156	157	157	157	157	158	159
162	163	163	164	166	166	168	170	174	196	212				

Table 3: Maximum likelihood estimates and goodness-of-fit statistics.

Model	Parameter estimates	$-\ell$	AIC	BIC
EOFNH	$\widehat{\alpha}=1.5838\ (0.2023)$ $\widehat{\beta}=1.1413\ (0.4131)$ $\widehat{\lambda}=0.0071\ (0.0041)$ $\widehat{\theta}=2.7505\ (0.1088)$	-470.3600	948.7255	959.1860
OFNH	$\widehat{\alpha}=0.4650\ (0.4204)$ $\widehat{\lambda}=0.0174\ (0.0263)$ $\widehat{\theta}=4.7456\ (1.8584)$	-473.6000	953.1958	961.0412
EOFW	$\widehat{\alpha}=2.2281\ (1.0985)$ $\widehat{\beta}=1.0205\ (0.3152)$ $\widehat{\lambda}=0.0099\ (0.0166)$ $\widehat{\theta}=2.3066\ (0.6716)$	-471.2600	950.5259	960.9864
OFW	$\widehat{\beta}=0.2785\ (0.0181)$ $\widehat{\lambda}=0.1823\ (0.0156)$ $\widehat{\theta}=13.2247\ (0.0007)$	-473.8200	953.6325	961.4778

The PDFs of the OFNH and OFW distributions are, respectively, given by

$$f(x) = \frac{\beta\lambda\theta(1+\lambda x)^{\beta-1} e^{(1-(1+\lambda x)^{\beta})}\left[1-\left(1-e^{1-(1+\lambda x)^{\beta}}\right)\right]^{\theta-1}}{\left(1-e^{(1-(1+\lambda x)^{\beta})}\right)^{\theta+1}} \cdot e^{-[(1-e^{(1-(1+\lambda x)^{\beta})})^{-1}-1]^{\theta}}, \quad x>0, \tag{44}$$

and

$$f(x) = \frac{\beta\lambda\theta x^{\beta-1} e^{-\lambda x^{\beta}}\left[1-\left(1-e^{-\lambda x^{\beta}}\right)\right]^{\theta-1}}{\left(1-e^{-\lambda x}\right)^{\theta+1}} \cdot e^{-[(1-e^{-\lambda x^{\beta}})^{-1}-1]^{\theta}}, \quad x>0. \tag{45}$$

Table 3 displays the maximum likelihood estimates of the parameters of the EOFNH, EOFW, OFNH, and OFW distributions with their corresponding standard errors in bracket and the model selection criteria. The results revealed that the EOFNH distribution provided the best fit for the data since it has the least values of AIC and the BIC. The EOFW distribution also performed better than the OFNH and OFW distributions. The OFNH distribution is a submodel of the EOFNH distribution with $\alpha = 1$. Hence, testing $H_0 : \alpha = 1$ versus $H_a : \alpha \neq 1$ using the LR test gave a test statistic of 6.4703 with corresponding $p-$value of 0.01097. This implies that there is enough evidence to reject H_0 at the 5% significance level and conclude that the EOFNH distribution provides better fit to the data than the OFNH distribution. Similarly, the LR test was performed to compare the performances of the EOFW distribution and the OFW distribution. The analysis gave a test statistic of 5.1065 with a corresponding $p-$value of 0.0238. This implies that the EOFW distribution performs better than the OFW distribution at the 5% significance level.

Figure 5 displays the histogram of the data with the fitted densities and the empirical CDF with the fitted CDFs.

The P-P plots of the fitted distributions are displayed in Figure 6.

8. Conclusion

The development of new statistical distribution plays a critical role in parametric statistical inference. Because of this, researchers in the field of distribution theory attempt to develop generators for generalizing the existing distributions. In line with this, the study developed and studied a new class of distributions called the EOF-G family. The statistical properties including the moments, incomplete

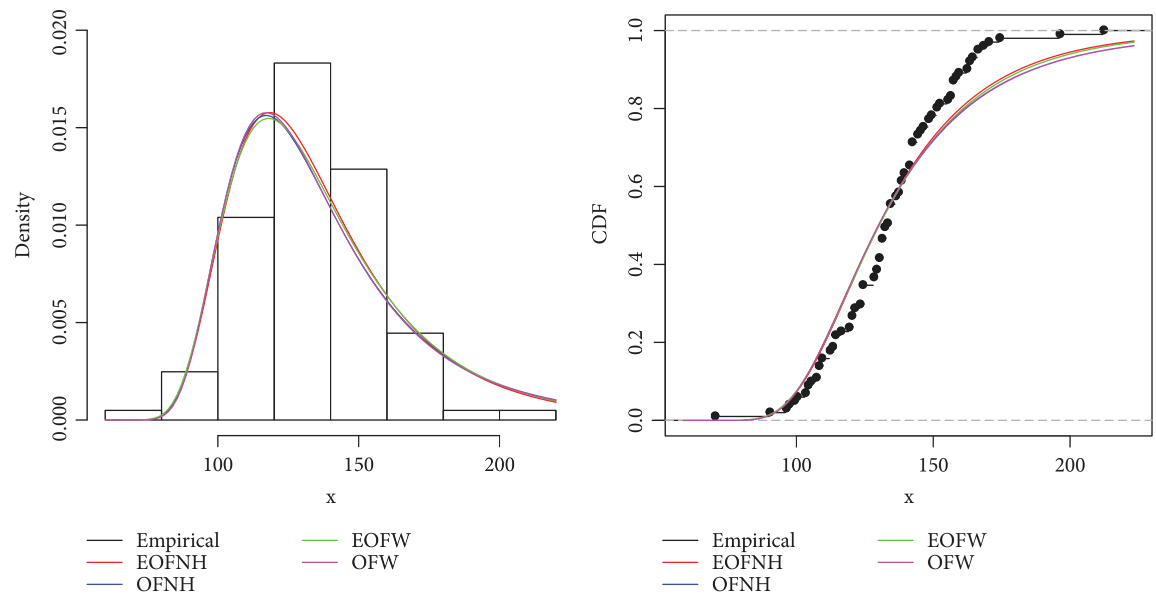

Figure 5: Plots of histogram of data and fitted densities; and empirical CDF and fitted CDF.

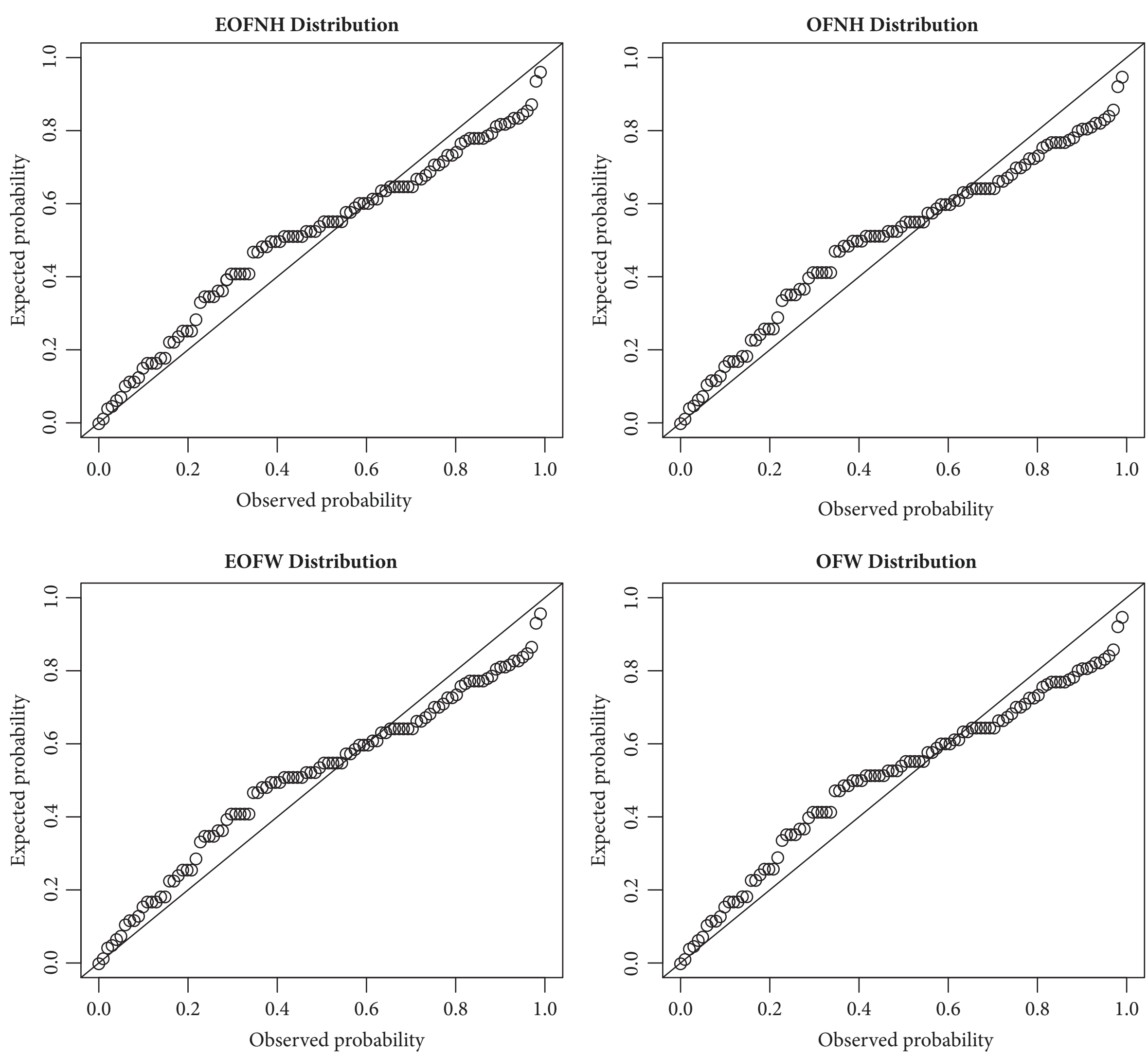

Figure 6: P-P plots of fitted distributions.

moments, generating function, entropies, and order statistics are derived. The maximum likelihood method is used to develop estimators for the parameters of the new family. The application of the special distributions developed using the EOF-G family is demonstrated using a real data set and the result compared with other existing distributions. From the application, it is evident that the special models developed from the EOF-G family can provide reasonable parametric fit to a given data set. Hence, it is hoped that the new class of distributions will attract wider applications in different fields of study.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

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