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Research Article

A Simple Empirical Likelihood Ratio Test for Normality Based on the Moment Constraints of a Half-Normal Distribution

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A simple and efficient empirical likelihood ratio (ELR) test for normality based on moment constraints of the half-normal distribution was developed. The proposed test can also be easily modified to test for departures from half-normality and is relatively simple to implement in various statistical packages with no ordering of observations required. Using Monte Carlo simulations, our test proved to be superior to other well-known existing goodness-of-fit (GoF) tests considered under symmetric alternative distributions for small to moderate sample sizes. A real data example revealed the robustness and applicability of the proposed test as well as its superiority in power over other common existing tests studied.

1. Introduction

Testing for distributional assumptions for normality is of paramount importance in applied statistical modelling. Several well-known numerical tests for normality are widely used by investigators to supplement the graphical techniques in assessing departures from normality. Amongst others, these tests include the Kolmogorov-Smirnov (KS) test [1], the Lilliefors (LL) test [2], the Anderson-Darling (AD) test [3, 4], the Shapiro-Wilks (SW) test [5], the Jarque-Bera (JB) test [6], and the DAgostino and Pearson (DP) test [7]. These tests differ on certain characteristics of the normal distribution on which they focus. That is, some focus on the empirical distribution function (EDF), some are moment based, and some are based on regression as well as correlation. Of these tests, some use normalized sample data whilst some use observed values. However, though these tests are commonly used in practice they do have major drawbacks. For example, some of these tests require complete specification of the null distribution, some require computation of critical values to be done for each specified null distribution, and some require ordering of the sample data when computing the test statistic. Generally, most of these tests are not supported when certain combinations of parameters of a specified distribution are estimated.

Of these, the most well-known goodness-of-fit (GoF) test is the SW test but it was originally restricted to small sample sizes (i.e., $n \leq 50$). Several modifications have been proposed by several researchers. These include Royston [8] who suggested a normalized transformation for the test in order to resolve the limitations on the sample size, Shapiro and Francia [9] who also modified the test so that it can be ideal for large sample sizes, Chen and Shapiro [10] who proposed normalized spacings for an alternative test of the SW test, and Rahman and Govindarajulu [11] who defined new weights for the SW test statistic. However, the major drawback of the SW test is computation time in dealing with large samples when computing the covariance matrix that corresponds to order statistics of the vector of weights and the standard normal distribution.

However, we also have GoF tests that are based on moment constraints such as the skewness and kurtosis coefficients and these are well known to be efficient tools for evaluating normality. These moment based tests include the skewness test, the kurtosis test, the DP test, and the JB test. These tests combine moment constraints to check for deviations from normality. They are often referred to as omnibus tests because of their ability to detect departures from normality whilst not depending upon the parameters of the normal distribution. The adoption of the use of moment based tests coupled with the empirical likelihood methodology has recently attracted the attention of researchers in developing GoF tests for normality [12, 13]. Dong and Giles [12] proposed an empirical likelihood ratio (ELR) test utilizing the empirical likelihood (EL) methodology of Owen [14]. They monitored the first four moment conditions of the normal distribution and their test outperformed alternate common existing tests studied against several alternative distributions. Our study followed from the works of Shan et al. [13] who proposed a simple ELR test for normality based on moment constraints using a standardized normal variable. Their test proved to be more powerful than other well-known GoF tests on small to moderate sample sizes for several alternative distributions. In this study we adopted their approach and focused on the construction of a simple ELR test for normality using the moment constraints of the half-normal distribution. The next section will outline the development of our proposed test followed by Monte Carlo simulations. A real data example will be presented. Discussions and conclusion of the findings as well as potential areas of future research will be highlighted.

2. ELR Test Development

Let us assume we have independent and identically distributed (*i.i.d*) nonordered random variables $X_1, X_2, \ldots, X_n$. The intention being to assess whether the observed data is normally distributed. Thus we intend testing the following null hypothesis:

$$H_0 : X \sim N\left(\mu, \sigma^2\right), \tag{1}$$

where μ and σ^2 are considered to be unknown parameters. We proposed using the standardized random variables of the normal distribution by using the following transformations:

$$Z_i^{\star} = \frac{X_i - \mu}{SD}, \quad i = 1, 2, \ldots, n, \tag{2}$$

where $\mu = \overline{X} = (1/n)\sum_{i=1}^{n} X_i$ and SD is the standard deviation to be estimated by an unbiased quantity $s^2 = S/(n-1)$. One can also decide to use the maximum likelihood estimate (MLE) $\widehat{\sigma}^2 = S/n$, where $S = \sum_{i=1}^{n}(X_i - \overline{X})^2$ and $\overline{X} = (1/n)\sum_{i=1}^{n} X_i$. Both quantities s^2 and $\widehat{\sigma}^2$ are known to converge to σ^2 as n approaches ∞. We also used an alternative transformation following Lin and Mudholkar's [17] work which also eliminates the dependency that exists between μ and σ on the data distribution. Thus we also transformed our observations using

$$Z_i^{*} = \frac{\sqrt{n/(n-1)}\left(X_i - \overline{X}\right)}{SD_{-i}}, \quad i = 1, 2, \ldots, n, \tag{3}$$

where $\overline{X} = (1/n)\sum_{j=1}^{n} X_j$, $SD_{-i}^2 = (1/(n-2))\sum_{j=1, j\neq i}^{n}(X_j - \overline{X}_{-i})^2$, and $\overline{X}_{-i} = (1/(n-1))\sum_{j=1, j\neq i}^{n} X_j$. As n gets large the standardized data points $Z_1, Z_2, \ldots, Z_n$ become asymptotically independent. If $X \sim N(0, \sigma^2)$, then the absolute value $|X| \sim HN(\mu, \sigma^2)$. It also follows that if $X \sim N(\mu, \sigma^2)$, then the modulus of the standardized normal random variables, $Z^{\star}$ and Z^{*}, follows a standardized half-normal random variable with mean = $\sqrt{2/\pi}$ and variance = 1. The standardized form of the half-normal distribution is also known as the χ^2-distribution with $\nu = 1$. The standardized half-normal random variable has a PDF that is given by

$$f_Z(z) = \begin{cases} \dfrac{2}{\sqrt{2\pi}} e^{-(1/2)z^2} & \text{for } z \geq 0, \\ 0 & \text{for } z < 0. \end{cases} \tag{4}$$

and we denote it as $Z \sim HN(\mu, \sigma^2)$. Following Prudnikov et al. [18], the k^{th} moment of the standardized half-normal variable for some integer $k > 0$ is as outlined in the proposition below.

Proposition 1. *Let $Z \sim HN(\sqrt{2/\pi}, 1)$, for $k = 1, 2, \ldots, n$, and then the k^{th} moments are given by*

$$E\left(Z^k\right) = \mu_k = \frac{1}{\sqrt{\pi}} 2^{k/2} \Gamma\left(\frac{k+1}{2}\right), \tag{5}$$

where $\Gamma(\cdot)$ denotes the gamma function.

We then derived the first four moments using the function given in (5). These moments are easily obtained as follows.

Corollary 2. *Let $Z \sim HN(\mu, \sigma^2)$. The first two moments of Z, that is μ and σ are given by*

$$E(Z) = \mu = \sqrt{\frac{2}{\pi}}\Gamma(1) = \sqrt{\frac{2}{\pi}} \approx 0.7979, \tag{6}$$

$$\operatorname{var}(Z) = \frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}\right) = 1. \tag{7}$$

Corollary 3. *Let $Z \sim HN(\mu, \sigma^2)$. The skewness and kurtosis coefficients of Z are given by*

$$\gamma(Z_1) = E\left(Z^3\right) = \mu_3 = \sqrt{\frac{2^3}{\pi}}\Gamma(2) = 2\sqrt{\frac{2}{\pi}} \approx 1.5958, \tag{8}$$

$$\gamma(Z_2) = E\left(Z^4\right) = \mu_4 = \frac{4}{\sqrt{\pi}}\Gamma\left(\frac{5}{2}\right) = 3. \tag{9}$$

In this study we used the first four moment constraints of the standardized half-normal distribution.

2.1. The ELR Based Test Statistic. We used an empirical likelihood ratio test (ELR) to construct our test statistic. Our aim was to compare the GoF test under H_0 against the alternative (H_a). In order to achieve this, we constructed our test statistic as follows. Let us consider n nonordered observations $X_1, X_2, \ldots, X_n$ that are independent and identically distributed and assumed to have unknown μ and σ. The intention is to perform a GoF test for the distributional assumption that $X_1, X_2, \ldots, X_n$ are consistent with a normal distribution. Now consider that the random variables $Z_1, Z_2, \ldots, Z_n$ are absolute standardized normal variables from the random variables $X_1, X_2, \ldots, X_n$. Thus the transformed/standardized observations have a moment function given in Proposition 1 above. Following the EL methodology we assigned p_i, which is a probability parameter to each transformed observation Z_i, and then formulated the EL function that is given by

$$L(F) = \prod_{i=1}^{n} p_i, \tag{10}$$

where p_i's satisfy the fundamental properties of probability; that is $0 \le p_i \le 1$ and $\sum_{i=1}^{n} p_i = 1$. Probability parameters, p_i's, will then be chosen subject to unbiased moment conditions and the EL method will utilize these p_i's in order to maximize the EL function. Following this EL technique, $E(Z^k)$ has sample moments $\sum_{i=1}^{n} p_i Z_i^k$ and the probability parameters (p_i's) are elements of the EL function. Under H_0, the four unbiased empirical moment equations have the form

$$\sum_{i=1}^{n} p_i Z_i^k - \mu_k = 0, \quad k = 1, 2, \ldots, n. \tag{11}$$

The composite hypotheses for the ELR test are given by

$$\begin{aligned} H_0 &: z_i' s \sim HN\left(\mu, \sigma^2\right) \\ \text{vs } H_a &: z_i' s \nsim HN\left(\mu, \sigma^2\right). \end{aligned} \tag{12}$$

Alternatively considering the above unbiased empirical moment equations, the hypotheses for the ELR test can be written as

$$\begin{aligned} H_0 &: E\left(Z^k\right) = \mu_k \\ \text{vs } H_a &: E\left(Z^k\right) \neq \mu_k, \end{aligned} \tag{13}$$

The nonparametric empirical likelihood function corresponding to the given hypotheses has the form:

$$L(F) = L\left(Z_1, Z_2, \ldots, Z_n \mid \mu_k\right) = \prod_{i=1}^{n} p_i, \tag{14}$$

where the unknown probability parameters and p_i's are attained under H_0 and H_a. Under H_0 the EL function is maximized with respect to the p_i's subject to two constraints

$$\begin{aligned} \sum_{i=1}^{n} p_i &= 1, \\ \sum_{i=1}^{n} p_i Z_i^k &= \mu_k. \end{aligned} \tag{15}$$

Following this, the weights of p_i's are identified as

$$\begin{aligned} p_1, p_2, \ldots, p_n = \sup \prod_{i=1}^{n} a_i \mid \quad & \sum_{i=1}^{n} a_i = 1, \\ \sum_{i=1}^{n} a_i Z_i^k = \mu_k, & \end{aligned} \tag{16}$$

where $0 \le a_j \le 1$, for $j = \{1, 2, \ldots, n\}$. If we then use the Lagrangian multipliers technique, it can be shown that the maximum EL function under H_0 can be expressed by the given form:

$$\begin{aligned} L\left(F_{H_0}\right) &= L\left(Z_1, Z_2, \ldots, Z_n \mid \mu_k\right) \\ &= \prod_{i=1}^{n} \frac{1}{n\left(1 + \lambda_k\left(Z_i^k - \mu_k\right)\right)}, \end{aligned} \tag{17}$$

where λ_k is a root of

$$\sum_{i=1}^{n} \frac{\left(Z_i^k - \mu_k\right)}{1 + \lambda_k\left(Z_i^k - \mu_k\right)} = 0. \tag{18}$$

Under the alternative hypothesis, $\sum_{i=1}^{n} p_i Z_i^k = \mu_k$ is not required to identify the weights, p_i, in order to maximize the EL function but only $\sum_{i=1}^{n} p_i = 1$. Thus under H_a the nonparametric EL function is given by

$$L\left(F_{H_a}\right) = L\left(Z_1, Z_2, \ldots, Z_n\right) = \prod_{i=1}^{n} \left(\frac{1}{n}\right) = \left(\frac{1}{n}\right)^n. \tag{19}$$

Now let us consider $(-2LLR)_k$ to be -2 log likelihood test statistic for the hypotheses $H_0 : E(Z^k) = \mu_k$ vs $H_a : E(Z^k) \neq \mu_k$. It should be noted that, under H_0, minus two times the log ELR has an asymymptotic χ^2 limiting distribution [19]. Thus considering the null and alternative hypotheses, the above test statistic will simply be transformed to

$$\begin{aligned} (-2LLR)_k &= -2 \log \frac{L\left(F_{H_0}\right)}{L\left(F_{H_a}\right)} \\ &= -2 \log \frac{L\left(Z_1, Z_2, \ldots, Z_n \mid \mu_k\right)}{L\left(Z_1, Z_2, \ldots, Z_n\right)}. \end{aligned} \tag{20}$$

With simple substitution the above can be simplified to

$$\begin{aligned} (-2LLR)_k &= -2 \log \frac{\prod_{i=1}^{n} \left(1/n\left(1 + \lambda_k\left(Z_i^k - \mu_k\right)\right)\right)}{\prod_{i=1}^{n} (1/n)} \\ &= 2 \sum_{i=1}^{n} \log\left[1 + \lambda_k\left(Z_i^k - \mu_k\right)\right]. \end{aligned} \tag{21}$$

We used the likelihood ratio to compare to size adjusted critical values in order to decide whether or not to reject H_0. We then proposed to reject the null hypothesis if

$$\mathrm{ELR}_Z = \max_{k \in G} (-2LLR)_k > C_\alpha, \tag{22}$$

where C_α is the test threshold and is $100(1-\alpha)\%$ percentile of the $\chi^2(1)$ distribution whilst G are integer values representing the set of moment constraints that maximizes the test statistic. As recommended by Dong and Giles [12], we used the first four moment constraints; that is, we set $G = \{1, 2, 3, 4\}$. In this study we used the abbreviation $\mathbf{ELR}_{Z1}$ to refer to the first test where we transformed data using (2) and we used the abbreviation $\mathbf{ELR}_{Z2}$ to refer to the second alternative test where we transformed data using (3). Our test statistic $\text{ELR}_Z = \max_{k\in G}(-2LLR)_k$ is a CUSUM-type statistic as classified by Vexler and Wu [20]. In their article, Vexler and Wu [20] stated that based on the change point literature, another common alternative is to utilize the Shiryaev-Roberts (SR) statistic in replacement of the CUSUM-type statistic (see, for example, [21, 22]). In our case the classical SR statistic was of the form $\sum_{k\in G}\exp(-2LLR)_k$. Vexler, Liu, and Pollak [23] showed that the classical SR statistic and the simple CUSUM-type statistic have almost equivalent optimal statistical properties due to their common null-martingale basis. Moreover, the classical SR statistic is adapted from the CUSUM-type statistic.

Shan et al. [13] used Monte Carlo experiments to compare the CUSUM-type statistic for their ELR test for normality with an equivalent classical SR statistic and based on the relative simplicity of the CUSUM-type statistic, as well as its power properties, the authors opted to use the CUSUM-type statistic for their study. We conducted a numerical experiment to compare power for the CUSUM-type and SR statistic for our proposed test statistics with increased moment constraints and, based on the same reasons given by Shan et al. [13], we decided to use the CUSUM-type statistic for our Monte Carlo comparisons. Also, from the results, ELR_{Z2} outperformed ELR_{Z1}, hence ELR_{Z2} was our preferred test. For all further comparisons, ELR_{Z1} was excluded in this study. Findings for this Monte Carlo experiment are presented in Table 4. However, it should be noted from these findings that ELR_{Z1} has the potential to be superior to ELR_{Z2} under certain alternatives. Further investigations to uncover the alternatives in which ELR_{Z1} is superior to ELR_{Z2} are a potential area of future research which will not be further addressed in this study. The next section will outline the Monte Carlo simulation procedures using the R statistical package.

3. Monte Carlo Simulation Study

We used the R statistical package to implement our Monte Carlo simulation procedures in power comparisons as well as assessment of our preferred proposed test (ELR_{Z2}). It should be noted that other standard statistical packages can easily be used to implement our proposed tests. In order for us to conduct any assessments and evaluations of the proposed test, firstly we had to determine the size adjusted critical values.

3.1. Size Adjusted Critical Values. Since the proposed ELR test is an asymptotic test, we therefore computed the unknown actual sizes for finite samples using Monte Carlo simulations with 50,000 replications. Motivated by practical applications, we considered critical values for relatively small sample sizes, i.e., $10 \leq n \leq 200$ because most applied statistical sciences datasets fall within this range. The actual rejection rate for a given sample size (n) is considered to be the total number of the rejections divided by the total number of replications. Data was simulated from a standard normal distribution. The stored ordered test statistics were then used to determine the percentiles of the empirical distribution. This makes it possible to obtain the 30%, 25%, 20%, . . . , 1%, size adjusted critical values.

3.2. ELR Test Assessment. The power of the proposed test (ELR_{Z2}) was compared to that of common existing GoF tests that include the Anderson-Darling (AD) test [3, 4] test, the modified Kolmogorov-Smirnov (KS) test [2] the Cramer-von Mises (CVM) test [24–26], the Jarque-Bera (JB) test [6], the Shapiro-Wilk (SW) test [5], the density based empirical likelihood ratio based (DB) test [16], and the simple and exact empirical likelihood test based on moment relations (SEELR) [13] at the 5% significance level. Power simulations were done using 5,000 replications for all tests with varying sample sizes (n = 20, 30, 50 and 80) against different alternative distributions. We adopted alternative distributions used by Shan et al. [13] which covers a wide range of both symmetric and asymmetric applied distributions. To assess robustness and applicability of our proposed test (ELR_{Z2}), we conducted a bootstrap study using some real data.

4. Results of the Monte Carlo Simulations

This section presents the findings of the power comparisons for the different categories of the alternative distributions considered. The results of the power comparisons are presented in Tables 5–8. Under symmetric cases defined on $(-\infty,\infty)$ our new test ELR_{Z2} outperformed all other studied tests against the considered alternative distributions but slightly inferior to the JB test. For symmetric distributions defined on $(0,1)$ our proposed test (ELR_{Z2}) was comparable to the DB test and significantly outperformed other alternate tests studied. However, when the alternative is Beta (0.5, 0.5), the ELR_{Z2} test is comparable to the SW and SEELR tests whilst only outperforming the KS test, the CVM test and the JB test.

As for asymmetric distributions defined on $(0,\infty)$, the SW and SEELR are the most powerful tests and should be the preferred tests under these cases. The AD and DB tests are comparable and they performed better than the proposed test as well as the KS and CVM tests. Lastly, in the category of asymmetric alternative distributions defined on $(-\infty,\infty)$ the ELR_{Z2} test was comparable to the SEELR test at low sample sizes (i.e., $n = 20, 30$) for the non-central t-distributions. The SW test outperformed all the tests considered in this study under these asymmetric alternative distributions. For the ELR based tests only the SEELR test was comparable to the common existing tests studied, that is, the AD test, the KS_M test, the CVM test, and the JB test.

Overall, when considering all the normality tests with respect to all of the alternative distributions considered, it can be seen that, the JB, the ELR_{Z2} and the SW tests are generally the most powerful tests given symmetric alternatives defined on $(-\infty,\infty)$, whilst the DB and the ELR_{Z2} tests are the most powerful tests for symmetric alternatives defined on $(0,1)$. On the other hand, the SEELR and the SW tests are the most powerful tests for asymmetric alternatives defined on $(0,\infty)$, whereas, the JB and SW tests are the most powerful tests for asymmetric alternatives defined on $(-\infty,\infty)$.

Table 1: Comparisons of computational times (in seconds) for the studied tests.

Test	Replications	Elapsed	Relative	User.self	Sys.self
AD	5,000	1.14	2.000	1.14	0.00
CVM	5,000	0.82	1.439	0.81	0.00
DB	5,000	17.00	29.825	16.94	0.04
ELR_{Z2}	5,000	44.83	78.649	44.78	0.01
JB	5,000	252.64	443.228	252.60	0.00
KS_M	5,000	0.89	1.561	0.90	0.00
SEELR	5,000	45.42	79.684	45.41	0.00
SW	5,000	0.57	1.000	0.58	0.00

Table 2: The baby boom data.

Times between births (in min)														
59	14	37	62	68	2	15	9	157	27	37	2	55	86	14
4	40	36	47	9	61	1	26	13	28	77	26	45	25	18
29	15	38	2	2	19	27	14	13	19	54	70	28		

Note. Data appeared in the newspaper the Sunday Mail on December 21, 1997 [15].

It was of paramount importance for us to determine the computational cost of the new algorithms by focusing on the computation time of the proposed test as compared to that of the considered existing tests. To assess this, we used the R benchmark tool on a notebook installed with 64 Bit Windows 10 Home addition. Equipped with a 4th generation Intel Core i5-4210U processor which has a speed of 1.7 GHz cache and memory (RAM) of 4 GB PC3 DDR3L SDRAM, we set our simulations to 5,000 for each test with sample size set at $n = 80$. The results (see Table 1) show only a clear advantage of our proposed approach to that of the widely known JB test. Also from the results, our proposed methods are comparable to the SEELR test but inferior to the DB test. The SW, CVM, KS and AD tests are computationally more efficient in terms of time than the rest of the studied tests.

5. A Real Data Example

In this example we used baby boom data from an observational study with records of forty-four (44) babies born at a 24-hour hospital in Brisbane, Australia. We opted for this dataset because it can be used to demonstrate applicability of various statistical procedures to some common applied distributions which include the normal (by modelling the birth weights), the binomial (inferences in the number of boys/girls born), the geometric (by considering the number of births until a boy/girl is born), the Poisson (births per hour for each hour), and the exponential (inference on times between births). Recently, Miecznikowski et al. [27] used the baby boom dataset in a resampling study on the application of their ELR based goodness-of-fit test. For more information regarding this dataset one can refer to Dunn [28]. For our application we opted to make use of the exponential distribution; thus we were interested in inference on the times between births. Table 2 shows the times between births which were computed by taking the differences between successive times of birth after midnight of birth times.

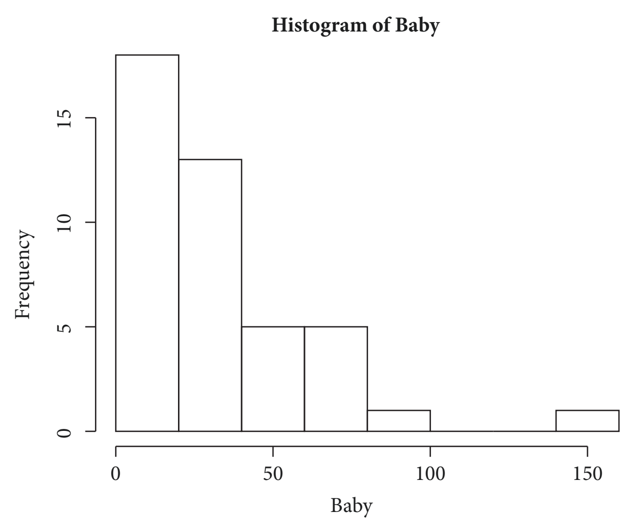

Figure 1: Histogram for times between births for baby boom data.

The goal of this example was to carry out a bootstrap study in assessing the robustness and applicability of our proposed test (ELR_{Z2}) on uniformly distributed data. However, the times between births are known to be consistent with the exponential distribution (see Figure 1). By assessing the histogram one can easily see that the data resembles the exponential distribution revealing that the times between births are exponentially consistent. We used the inverse exponential distribution to transform the times between births so that they can be uniformly distributed. We then used the density based empirical likelihood ratio based test (dbEmpLikeGOF) to check if the transformed baby boom data are uniformly distributed. The dbEmpLikeGOF test returned a p value of 0.6950 suggesting that the transformed data are consistent with the uniform distribution.

Table 3: Bootstrapping using the inverse exponential transformed baby boom data.

	Bootstrap power comparisons: H_0 : data is normally distributed							
Observations removed	AD	KS_M	CVM	JB	SW	DB	SEELR	ELR_{Z2}
3	0.0000	0.0000	0.0000	0.0000	0.1012	0.6040	0.0000	0.6132
8	0.0164	0.0006	0.0012	0.0000	0.1182	0.4486	0.0146	0.4054
13	0.0281	0.0073	0.0066	0.0011	0.1023	0.3488	0.0568	0.2714

Table 4: A numerical assessment on power using the Shiryaev-Roberts (S-R) and CUSUM-type (C-t) statistics for the proposed tests (ELR_{Z1} and ELR_{Z2}) with increased moment constraints at $\alpha = 0.05$.

	ELR_{Z1}				ELR_{Z2}			
n	$k = \{1, 2, 3, 4\}$		$k = \{1, 2, 3, 4, 5\}$		$k = \{1, 2, 3, 4\}$		$k = \{1, 2, 3, 4, 5\}$	
	S-R	C-t	S-R	C-t	S-R	C-t	S-R	C-t
				$t(2)$				
30	0.0416	0.0330	0.0020	0.0010	0.6980	**0.6998**	0.6166	0.5912
50	0.5142	0.4112	0.1666	0.1356	0.8766	**0.8774**	0.8320	0.8030
80	0.8732	0.8336	0.7476	0.7184	**0.9718**	0.9684	0.9544	0.9488
				Cauchy(0,1)				
30	0.3262	0.3438	0.0000	0.0000	**0.9560**	0.9556	0.9248	0.9192
50	0.9538	0.9344	0.7246	0.6754	0.9970	**0.9974**	0.9928	0.9900
80	0.9996	0.9996	0.9964	0.9940	**1.0000**	**1.0000**	0.9998	0.9996
				Uniform(0,1)				
30	**0.7230**	0.1958	0.7208	0.7206	0.5772	0.5986	0.6996	0.7004
50	0.9458	0.5222	**0.9532**	0.9474	0.9032	0.9122	0.9434	0.9398
80	0.9966	0.8462	0.9978	0.9980	0.9940	0.9956	**0.9986**	0.9976
				Exp(1)				
30	0.0094	0.0304	0.0070	0.0068	0.4638	**0.4818**	0.3874	0.3772
50	0.0836	**0.8096**	0.0022	0.0042	0.6274	0.6306	0.5628	0.5380
80	0.3764	**0.9972**	0.2504	0.2346	0.7942	0.8070	0.7558	0.7506
				$t(\delta = 1, \nu = 2)$				
30	0.0476	0.0136	0.0012	0.0028	0.7168	**0.7230**	0.6476	0.6280
50	0.5204	0.4172	0.1932	0.1676	0.8904	**0.8908**	0.8450	0.8294
80	0.8752	0.8610	0.7736	0.7700	0.9714	**0.9766**	0.9636	0.9558
				SN(0,1,5)				
30	0.0514	0.0520	0.0486	0.0442	**0.1394**	0.1242	0.1048	0.0944
50	0.0404	0.0362	0.0350	0.0352	0.1408	**0.1432**	0.1114	0.0904
80	0.0358	0.0338	0.0272	0.0204	0.1592	**0.1646**	0.1158	0.1226

Note. Our proposed tests are maximized on $k \in G$, where G can take any integer to represent the moment constraints used to maximise the test statistics for specified sample sizes at 5% level of significance using 5,000 simulations. n is the sample size. **Bold** represents the powerful test statistic for the given simulation scenarios.

For the resampling study we performed a power simulation study by randomly removing 3, 8, and 13 observations from the transformed baby boom data at 5% significance level using 20,000 replications for each simulation. For comparison's sake we considered the AD test, the modified KS test, the CVM test, the JB test, the SW test, the DB test, the SEELR test, and our proposed test (ELR_{Z2}). The Monte Carlo bootstrap simulation results are presented in Table 3. It is undeniably clear that our test outperformed all the common existing tests and therefore suggests its robustness and applicability on real data. It should be noted that we opted for uniformly distributed data for our application since our proposed test (ELR_{Z2}) proved to be more powerful for symmetric alternative distributions which are defined on $(0, 1)$.

6. Conclusion

An empirical likelihood ratio test for normality based on moment constraints of the half-normal distribution has been developed. Overall, the proposed ELR test has good power properties and significantly outperformed the considered well-known common existing tests against the studied alternative symmetric distributions. In our case, the attractive power properties of the proposed ELR test resulted from the EL method being able to integrate most of the available information by utilizing the first four moment constraints and also through the utilization of the EL function which leads to additional power benefits. We advocate for our proposed test (ELR_{Z2}) to be the preferred choice when one

Table 5: Results of the Monte Carlo power comparisons based on samples with sizes (n) from **symmetric alternative distributions** defined on $(-\infty, \infty)$ at $\alpha = 0.05$.

Symmetric alternative distributions defined on $(-\infty, \infty)$ at $\alpha = 0.05$									
Distribution	n	AD	KS_M	CVM	JB	SW	DB	SEELR	ELR_{Z2}
t(2)	20	0.5068	0.4482	0.5138	0.5632	0.5282	0.2806	0.3774	0.5268
	30	0.6834	0.5832	0.6552	0.7016	0.6908	0.3946	0.4228	0.7004
	50	0.8538	0.7782	0.8370	0.8812	0.8572	0.5640	0.4800	0.8726
	80	0.9602	0.9200	0.9554	0.9646	0.9566	0.8010	0.5420	0.9658
t(4)	20	0.2270	0.1768	0.2114	0.2898	0.2410	0.0922	0.1698	0.2450
	30	0.3002	0.2182	0.2764	0.3788	0.3338	0.1084	0.2164	0.3398
	50	0.4150	0.3176	0.3794	0.5400	0.4520	0.1388	0.2468	0.4784
	80	0.5558	0.3994	0.5210	0.7064	0.6282	0.2094	0.2784	0.6760
t(7)	20	0.1162	0.0952	0.1006	0.1670	0.1398	0.0492	0.1066	0.1346
	30	0.1404	0.1008	0.1306	0.2222	0.1806	0.0552	0.1188	0.1664
	50	0.1806	0.1272	0.1578	0.2954	0.2362	0.0502	0.1422	0.2276
	80	0.2380	0.1618	0.2086	0.4010	0.3122	0.0650	0.1590	0.3324
Cauchy(0,1)	20	0.8780	0.8386	0.8898	0.8622	0.8674	0.7012	0.6368	0.8450
	30	0.9672	0.9410	0.9622	0.9574	0.9610	0.8606	0.6910	0.9542
	50	0.9976	0.9950	0.9964	0.9954	0.9958	0.9712	0.7424	0.9976
	80	1.0000	1.0000	0.9998	0.9998	0.9998	0.9992	0.8882	1.0000
Cauchy(0,5)	20	0.8778	0.8374	0.8796	0.8650	0.8704	0.6902	0.6454	0.8550
	30	0.9628	0.9414	0.9648	0.9512	0.9590	0.8664	0.6950	0.9542
	50	0.9968	0.9948	0.9976	0.9968	0.9966	0.9730	0.7468	0.9962
	80	1.0000	1.0000	1.0000	0.9998	1.0000	0.9996	0.8872	1.0000
Logistic	20	0.1090	0.0872	0.0982	0.1460	0.1138	0.0436	0.0944	0.1158
	30	0.1176	0.0908	0.1220	0.1982	0.1474	0.0452	0.1044	0.1482
	50	0.1562	0.1184	0.1456	0.2620	0.1986	0.0414	0.1216	0.1900
	80	0.2098	0.1406	0.1870	0.3474	0.2662	0.0468	0.1266	0.2908

Anderson-Darling (**AD**) test, Modified Kolmogorov-Smirnov (**KS_M**) test [2], Cramer-von Mises test (**CVM**) test, Jarque-Bera (**JB**) test, Shapiro-Wilk (**SW**) test, density based empirical likelihood ratio based (**DB**) test [16], simple and exact empirical likelihood ratio based (**SEELR**) test [13], and the proposed test **ELR_{Z2}**.

Table 6: Results of the Monte Carlo power comparisons based on samples with sizes (n) from **symmetric alternative distributions** defined on $(0, 1)$ at $\alpha = 0.05$.

Symmetric alternative distributions defined on $(0, 1)$ at $\alpha = 0.05$									
Distribution	n	AD	KS_M	CVM	JB	SW	DB	SEELR	ELR_{Z2}
Beta(2,2)	20	0.0564	0.0544	0.0594	0.0052	0.0516	0.1310	0.0696	0.0970
	30	0.0786	0.0520	0.0812	0.0012	0.0768	0.2004	0.0550	0.1962
	50	0.1222	0.0852	0.1172	0.0010	0.1528	0.3468	0.0628	0.4252
	80	0.2340	0.1256	0.1834	0.0128	0.3170	0.5978	0.1128	0.7204
Beta(3,3)	20	0.0404	0.0474	0.0408	0.0076	0.0372	0.0780	0.0518	0.0620
	30	0.0786	0.0520	0.0812	0.0046	0.0768	0.1112	0.0392	0.1030
	50	0.0736	0.0524	0.0650	0.0014	0.0682	0.1654	0.0326	0.1906
	80	0.1076	0.0762	0.0826	0.0022	0.1128	0.2772	0.0298	0.3458
Beta(0.5,0.5)	20	0.6160	0.3098	0.5058	0.0066	0.7190	0.9094	0.7092	0.7015
	30	0.8576	0.4998	0.7332	0.0052	0.9392	0.9914	0.8830	0.8960
	50	0.9902	0.7976	0.9568	0.3822	0.9992	1.0000	0.9916	0.9956
	80	1.0000	0.9724	0.9990	0.9872	1.0000	1.0000	1.0000	1.0000
Uniform(0,1)	20	0.1640	0.1014	0.1396	0.0040	0.1886	0.4064	0.2598	0.3332
	30	0.3004	0.1422	0.2262	0.0020	0.3894	0.6622	0.3202	0.6002
	50	0.5780	0.2532	0.4282	0.0118	0.7546	0.9358	0.5624	0.9120
	80	0.8636	0.4578	0.7092	0.3706	0.9688	0.9990	0.8730	0.9944
Logit-norm(0,1)	20	0.0648	0.0442	0.0562	0.0056	0.0578	0.1294	0.0700	0.1010
	30	0.0858	0.0574	0.0748	0.0024	0.0796	0.1974	0.0658	0.1990
	50	0.1394	0.0812	0.1220	0.0010	0.1612	0.3420	0.0676	0.4156
	80	0.2630	0.1368	0.2114	0.0126	0.3408	0.5830	0.1094	0.7108
Logit-norm(0,2)	20	0.3758	0.1844	0.2934	0.0046	0.4366	0.7034	0.4806	0.5348
	30	0.6092	0.2884	0.4822	0.0030	0.7342	0.9150	0.6512	0.8258
	50	0.9016	0.5412	0.7814	0.1174	0.9742	0.9976	0.9006	0.9818
	80	0.9942	0.8170	0.9644	0.8594	1.0000	1.0000	0.9958	0.9996

Anderson-Darling (**AD**) test, Modified Kolmogorov-Smirnov (**KS_M**) test [2], Cramer-von Mises test (**CVM**) test, Jarque-Bera (**JB**) test, Shapiro-Wilk (**SW**) test, density based empirical likelihood ratio based (**DB**) test [16], simple and exact empirical likelihood ratio based (**SEELR**) test [13], and the proposed test **ELR_{Z2}**.

Table 7: Results of the Monte Carlo power comparisons based on samples with sizes (n) from **asymmetric alternative distributions** defined on $(0, \infty)$ at $\alpha = 0.05$.

Asymmetric alternative distributions defined on $(0, \infty)$ at $\alpha = 0.05$									
Distribution	n	AD	KS_M	CVM	JB	SW	DB	SEELR	ELR_{Z2}
Exp(1)	20	0.7850	0.5722	0.7222	0.6230	0.8334	0.8384	0.8522	0.3642
	30	0.9296	0.7780	0.8922	0.8286	0.9646	0.9754	0.9996	0.4752
	50	0.9972	0.9594	0.9878	0.9756	0.9998	0.9992	1.0000	0.6400
	80	1.0000	0.9990	0.9998	0.9998	1.0000	1.0000	1.0000	0.8114
Gamma(2,1)	20	0.4590	0.3066	0.4136	0.4080	0.5380	0.4420	0.5684	0.2264
	30	0.6662	0.4776	0.6072	0.5852	0.7502	0.6876	0.8094	0.2844
	50	0.8960	0.6926	0.8436	0.8242	0.9500	0.9180	0.9668	0.3822
	80	0.9840	0.8962	0.9682	0.9782	0.9976	0.9914	0.9984	0.5210
Lognorm(0,1)	20	0.9080	0.7760	0.8846	0.8172	0.9350	0.9210	0.9418	0.6036
	30	0.9838	0.9304	0.9730	0.9466	0.9888	0.9906	1.0000	0.7418
	50	1.0000	0.9942	0.9998	0.9976	1.0000	1.0000	1.0000	0.9068
	80	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9838
Lognorm(0,2)	20	0.9986	0.9904	0.9970	0.9840	0.9990	0.9998	0.9999	0.8894
	30	0.9998	0.9998	1.0000	0.9994	1.0000	1.0000	1.0000	0.9684
	50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9988
	80	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
Weibull(2,1)	20	0.1348	0.0980	0.1142	0.1258	0.1582	0.1264	0.1626	0.0932
	30	0.1828	0.1306	0.1654	0.1704	0.2274	0.1958	0.2718	0.0892
	50	0.3050	0.2000	0.2530	0.2738	0.4086	0.3446	0.5202	0.1120
	80	0.4954	0.3186	0.4200	0.4346	0.6644	0.5634	0.7812	0.1080
Weibull(0.5,1)	20	0.9962	0.9810	0.9954	0.9562	0.9990	0.9996	0.9986	0.8014
	30	1.0000	0.9990	1.0000	0.9972	1.0000	1.0000	1.0000	0.9168
	50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9866
	80	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Anderson-Darling (**AD**) test, Modified Kolmogorov-Smirnov (**KS_M**) test [2], Cramer-von Mises test (**CVM**) test, Jarque-Bera (**JB**) test, Shapiro-Wilk (**SW**) test, density based empirical likelihood ratio based (**DB**) test [16], simple and exact empirical likelihood ratio based (**SEELR**) test [13], and the proposed test **ELR_{Z2}**.

Table 8: Results of the Monte Carlo power comparisons based on samples with sizes (n) from **asymmetric alternative distributions** defined on $(-\infty, \infty)$ at $\alpha = 0.05$.

Asymmetric alternative distributions defined on $(-\infty, \infty)$ at $\alpha = 0.05$									
Distribution	n	AD	KS_M	CVM	JB	SW	DB	SEELR	ELR_{Z2}
$t(\delta = 1, \nu = 2)$	20	0.6446	0.5692	0.6440	0.6556	0.6498	0.4612	0.5688	0.5542
	30	0.8060	0.7178	0.7934	0.8080	0.8072	0.6210	0.6678	0.7242
	50	0.9492	0.8900	0.9394	0.9414	0.9410	0.7820	0.7782	0.8872
	80	0.9928	0.9762	0.9892	0.9924	0.9924	0.9294	0.8410	0.9726
$t(\delta = 1, \nu = 4)$	20	0.3180	0.2368	0.2848	0.3606	0.3142	0.1638	0.2790	0.2744
	30	0.4086	0.3246	0.3884	0.4810	0.4518	0.2262	0.3584	0.3718
	50	0.5912	0.4618	0.5538	0.6592	0.6360	0.3202	0.4830	0.5344
	80	0.7642	0.6370	0.7296	0.8290	0.8108	0.4626	0.5826	0.7084
$t(\delta = 1, \nu = 7)$	20	0.1490	0.1138	0.1404	0.1934	0.1692	0.0766	0.1420	0.1492
	30	0.1958	0.1424	0.1736	0.2722	0.2318	0.0940	0.1876	0.1936
	50	0.2846	0.1968	0.2522	0.3834	0.3372	0.1194	0.2556	0.2738
	80	0.3848	0.2756	0.3542	0.5102	0.4556	0.1620	0.3300	0.3798
SN(0,1,2)	20	0.0896	0.0756	0.0882	0.1054	0.1068	0.0636	0.0978	0.0710
	30	0.1194	0.0912	0.1062	0.1214	0.1422	0.0784	0.1336	0.0792
	50	0.1666	0.1260	0.1488	0.1810	0.1968	0.1164	0.2080	0.0800
	80	0.2434	0.1918	0.2258	0.2712	0.2940	0.1550	0.3074	0.0890
SN(0,1,5)	20	0.2406	0.1744	0.2212	0.2060	0.2660	0.2092	0.2810	0.1144
	30	0.3586	0.2604	0.3152	0.3056	0.4230	0.3346	0.4742	0.1278
	50	0.5796	0.4098	0.5430	0.4768	0.6672	0.5250	0.7110	0.1372
	80	0.8080	0.6084	0.7554	0.7378	0.8888	0.7394	0.9020	0.1722
SC(0,2,5)	20	0.9660	0.9360	0.9658	0.9410	0.9736	0.9436	0.9462	0.8482
	30	0.9978	0.9884	0.9954	0.9910	0.9970	0.9882	0.9774	0.9524
	50	1.0000	0.9998	1.0000	0.9998	0.9998	0.9992	0.9844	0.9940
	80	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9976	0.9998

Anderson-Darling (**AD**) test, Modified Kolmogorov-Smirnov (**KS_M**) test [2], Cramer-von Mises test (**CVM**) test, Jarque-Bera (**JB**) test, Shapiro-Wilk (**SW**) test, density based empirical likelihood ratio based (**DB**) test [16], simple and exact empirical likelihood ratio based (**SEELR**) test [13], and the proposed test **ELR_{Z2}**.

is testing for departures from normality against symmetric alternative distributions for small to moderate sample sizes. However, our test has low power in the considered asymmetric alternatives and further modifications in improving the power of the test under these alternatives would be much appreciated.

In this study we used the moment constraints of the standardized variables of the half-normal distribution. It will be of interest for one to use the raw moments (nonstandardized data points) of the half-normal distribution. However, according to Dong and Giles [12], the power of the ELR test using standardized observations is within the same range as it is when using nonstandardized data points. Also of interest are the findings by Mittelhammer et al. [29] where they suggested that the power of ELR based tests increases as the moment constraints increase. From our numerical experiment we did not extensively explore this conjecture and this is a potential area of future research and it might be interesting to carry out a more detailed investigation for the proposed tests. We focused on tests for normality, which is a common distribution to test in applied statistical modelling and we believe that our proposed test will assist investigators to use empirical likelihood approaches using moment constraints for goodness-of-fit tests of other applied distributions in practice. By simply ignoring the absolute values of the transformed observations and utilizing standardized half-normal data points our proposed test will simply transform to a GoF test for assessing departures from half-normality.

Data Availability

The data appeared in an article entitled “Babies by the Dozen for Christmas: 24-Hour Baby Boom” in the newspaper the Sunday Mail on December 21, 1997 [15]. One can get the data in the package ‘dbEmpLikeGOF’ in R.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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