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Research Article

An Empirical Likelihood Ratio-Based Omnibus Test for Normality with an Adjustment for Symmetric Alternatives

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An omnibus test for normality with an adjustment for symmetric alternatives is developed using the empirical likelihood ratio technique. We first transform the raw data via a jackknife transformation technique by deleting one observation at a time. The probability integral transformation was then applied on the transformed data, and under the null hypothesis, the transformed data have a limiting uniform distribution, reducing testing for normality to testing for uniformity. Employing the empirical likelihood technique, we show that the test statistic has a chi-square limiting distribution. We also demonstrated that, under the established symmetric settings, the CUSUM-type and Shiryaev–Roberts test statistics gave comparable properties and power. The proposed test has good control of type I error. Monte Carlo simulations revealed that the proposed test outperformed studied classical existing tests under symmetric short-tailed alternatives. Findings from a real data study further revealed the robustness and applicability of the proposed test in practice.

1. Introduction

The empirical likelihood (EL) methodology was introduced in [1, 2] and has been widely studied as a nonparametric approximation of the parametric likelihood approach (e.g., [3–6]). Thus, it utilizes the concept of the likelihoods in a distribution-free manner in approximating optimal parametric likelihood-based techniques. The method provides a versatile approach that may be applied to perform inference for a wide variety of statistical applications. An area with substantial new development in the use of the EL methods is hypothesis testing. Various researchers have proposed goodness-of-fit (GoF) tests for continuous distributions based on the EL for a wide range of hypothesis tests, which includes exponentiality [7, 8], logistic [9], uniformity [10], and normality [11, 12].

From the various proposed EL testing procedures as well as in the current statistical practice, it is evident that the problem of testing composite hypotheses of normality is undeniably the most common research focus in GoF testing. The continued growing need for normality tests is attributed to the frequent use and applications of normally distributed data in various areas of pure and applied statistical practices. Although it is difficult to propose a test for normality competing with the highly efficient family of Shapiro–Wilk tests (e.g., [13–16]), the proposed EL-based normality tests have proved to be superior under certain alternative distributions [12]. Of these tests, the moment-based tests seem to have gained more traction due to their flexibility, simplicity, power properties, and convenient use of omnibus tests in accessing the normality of underlying continuous distributions.

To test for normality, Dong and Giles [11] proposed an omnibus test statistic by directly utilizing the EL methodology outlined by Owen [17]. They utilized the first four moment constraints that characterize the normal distribution. After outlining drawbacks of the test proposed by Dong and Giles [11], Shan et al. [12] proposed a cumulative sum-

(CUSUM-) type simple and exact empirical likelihood ratio-based (SEELR) test statistic for normality which unlike that of Dong and Giles [11] has good control of type I error (also see [18]) and can be easily implemented in a wide range of statistical packages. The test by Shan et al. [12] is an omnibus test that makes use of standardized sample observations using the Lin and Mudholkar [19] jackknife transformation. In their study, Shan et al. [12] reported that power of their proposed omnibus test is comparable to well-known existing tests and oftentimes outperforms these tests under certain alternatives, mostly asymmetric distributions.

Just like some tests for normality, the test proposed by Shan et al. [12] suffers the loss of power under several symmetric alternatives. It is a challenge to propose an omnibus test that has high power than the classical Jarque–Bera's tests [20, 21] and the D'Agostino–Pearson k^2 test [22] in detecting departures from normality in alternatives that exhibit the symmetric nature of the normal distribution. Through the utilization of various mathematical and statistical properties that characterize the normal distribution (for example, see [19, 23, 24]) one can remedy such shortfalls in GoF tests. One such remedy is transformation to uniformity, which has several benefits that include increasing the power of a test under certain alternatives (for example, see [8, 25]). For a data-driven omnibus test for symmetry, Fang et al. [25] utilized a bootstrapping approach coupled with the probability integral transformation, and under the null hypothesis, the transformed data had a limiting uniform distribution. For superior power under symmetric alternatives, their proposed test required only odd-ordered orthogonal moments of the transformed data in constructing the test statistic.

The use of the probability integral transformation in the development of GoF tests of normality has been widely used especially in empirical distribution function- (EDF-) based tests. Rosenblatt [26] first introduced the concept. Thus, the EDF tests make use of the probability integral transformation $U = F(X)$. If $F(X)$ is the distribution function of X, the random variable U is uniformly distributed between 0 and 1. Given n observations $X_{(1)}, \ldots, X_{(n)}$, the values $U_{(i)} = F(X_{(i)})$ are computed. The most commonly used EDF tests for normality are the Anderson–Darling test [27, 28] and the Lilliefors test which is well known as the modified Kolmogorov–Smirnov test [29] and the Cramér–von Mises test [30]. In addition to the use of the probability integral transformation, several approaches have been used to construct GoF tests for the composite hypothesis of normality. In this study, we adopted the EL methodology to propose a new omnibus test for normality by exploiting different forms of characterizing the normal distribution. The purpose of this paper is to use a jackknife characterization due to Shan et al. [12], as in Lin and Mudholkar [19], followed by a probability integral transformation (see [8, 25]) for developing a goodness-of-fit test for normality. Here we consider the approach to obtaining a GoF test statistic by combining two well-known characterizations, individually powerful against different classes of alternatives. However, following the works of Fang et al. [25], we restrict attention to symmetric alternatives. Power comparisons are conducted with some of the most widely known EDF-based tests, well-known and powerful moment-based tests, and the powerful classical SW tests.

2. Test Development

Consider an unknown continuous distribution with non-ordered random variables denoted by $X_1, X_2, \ldots, X_n$ that are assumed to be independent and identically distributed (*i.i.d.*). The intention is to test whether the observations are consistent with a normal distribution. Thus, we intended to test whether to accept or reject the following null hypothesis:

$$H_0\colon X_1, X_2, \ldots, X_n \sim N(\mu, \sigma^2), \tag{1}$$

where μ and σ^2 are unknown parameters. We then proposed to use standardized random variables of the sample observations. To achieve this, we adopted a jackknife transformation technique by deleting one observation at a time following Lin and Mudholkar [19] works (also see [12]). Thus, we transformed our observations using

$$Z_i = \frac{\sqrt{(n/n-1)}\,(X_i - \overline{X})}{\mathrm{SD}_{-i}}, \quad i = 1, 2, \ldots, n, \tag{2}$$

where $\overline{X} = (1/n)\sum_{j=1}^{n} X_j$, $\mathrm{SD}_{-i}^2 = (1/n-2)\sum_{j=1, j\neq i}^{n} (X_j - \overline{X}_{-i})^2$, and $\overline{X}_{-i} = (1/n-1)\sum_{j=1, j\neq i}^{n} X_j$. It should be noted according to Shan et al. [12] as n gets large, the standardized data points $Z_1, Z_2, \ldots, Z_n$ become asymptotically independent while under the null hypothesis they are distributed according to a t distribution with $n-2$ df, which as n grows approaches the standard normal. In addition to this transformation, we then further adopted the probability integral transformation (see [25] as well as [8]). The probability integral transformation then transformed the standardized random variables into independent uniformly distributed random variables $Y_1, Y_2, \ldots, Y_n$. That is, under the null hypothesis, the transformed data follow the uniform distribution asymptotically. From the proposed transformation, Y_i are uniformly distributed on $(0, 1)$, where the density function of $Y(a, b)$ is given by

$$f(y) = \begin{cases} \dfrac{1}{b-a}, & \text{for } a \le y \le b, \\ 0, & \text{otherwise}, \end{cases} \tag{3}$$

where a = the lowest value of y and b = the highest value of y. The k^{th} moment of the uniform distribution is defined by

$$E(Y^k) = \frac{b^{k+1} - a^{k+1}}{(k+1)(b-a)}. \tag{4}$$

From the uniformly transformed observations, we then proposed to test for the following null hypothesis:

$$H_0\colon Y_1, Y_2, \ldots, Y_n \sim \mathrm{Uniform}(0, 1), \tag{5}$$

versus the alternative that $Y_1, Y_2, \ldots, Y_n$ are from a non-uniform distribution defined on (0, 1). In accordance with the EL method, under H_0, we considered unbiased empirical moment equations utilizing the raw moments of the uniform distribution, which are given by

$$\sum_{i=1}^{n} p_i Y_i^k - \mu_k = 0, \quad k = 1, 2, \ldots, m. \tag{6}$$

The composite hypothesis for the ELR test was then given by

$$H_0\colon E\left(Y^k\right) = \mu_k \text{ vs } H_a\colon E\left(Y^k\right) \neq \mu_k, \quad k = 1, 2, \ldots, m. \tag{7}$$

The nonparametric empirical likelihood function corresponding to the given hypotheses in equation (7) is expressed as

$$L(F) = L\left(Y_1, Y_2, \ldots, Y_n \mid \mu_k\right) = \prod_{i=1}^{n} p_i, \tag{8}$$

where the unknown probability parameters p_i's are attained under H_0 and H_a. Under H_0, the EL function given in equation (8) is maximized with respect to the p_i's subject to two constraints:

$$\begin{aligned} &\sum_{i=1}^{n} p_i = 1, \\ &\sum_{i=1}^{n} p_i Y_i^k = \mu_k, \quad k = 1, 2, \ldots, m. \end{aligned} \tag{9}$$

Following this, the weights of p_i's are identified as

$$p_1, p_2, \ldots, p_n = \arg \max_{a_1, a_2, \ldots, a_n} \prod_{i=1}^{n} a_i \left| \sum_{i=1}^{n} a_i = 1, \ \sum_{i=1}^{n} a_i Y_i^k = \mu_k, \right. \tag{10}$$

where $0 \leq a_j \leq 1$, for $j = 1, 2, \ldots, n$. If we then use the Lagrangian multipliers technique, it can be shown that the maximum EL function under H_0 can be expressed as

$$L\left(F_{H_0}\right) = L\left(Y_1, Y_2, \ldots, Y_n \mid \mu_k\right) = \prod_{i=1}^{n} \frac{1}{n\left(1 + \lambda_k\left(Y_i^k - \mu_k\right)\right)}, \tag{11}$$

where λ_k in equation (11) is a root of

$$\sum_{i=1}^{n} \frac{\left(Y_i^k - \mu_k\right)}{1 + \lambda_k\left(Y_i^k - \mu_k\right)} = 0. \tag{12}$$

Under the alternative hypothesis, $\sum_{i=1}^{n} p_i Y_i^k = \mu_k$ is not required to identify the weights, p_i in order to maximize the EL function but only, $\sum_{i=1}^{n} p_i = 1$. Thus, under H_a, the nonparametric EL function is given by

$$L\left(F_{H_a}\right) = L\left(Y_1, Y_2, \ldots, Y_n\right) = \prod_{i=1}^{n} \left(\frac{1}{n}\right) = \left(\frac{1}{n}\right)^n. \tag{13}$$

Now let us consider $(-2\text{LLR})_k$ to be -2 log-likelihood test statistic for the hypotheses H_0: $E(Y^k) = \mu_k$ vs. H_a: $E(Y^k) \neq \mu_k$. It should be noted that, under H_0, minus two times the log LLR has an asymptotic $\chi^2(1)$ limiting distribution [1]. Considering the null and alternative hypotheses, the test statistic is given by

$$(-2\text{LLR})_k = -2 \log \frac{L\left(F_{H_0}\right)}{L\left(F_{H_a}\right)} = 2 \sum_{i=1}^{n} \log\left[1 + \lambda_k\left(Y_i^k - \mu_k\right)\right]. \tag{14}$$

We then proposed to reject the null hypothesis using two different test statistics. Firstly, we considered the cumulative sum- (CUSUM-) type statistic given by

$$\text{CSELR} = \max_{k \in G} \, (-2\text{LLR})_k > C_\alpha. \tag{15}$$

Secondly, we considered the common alternative to the CUSUM-type statistic, which is to utilize the Shiryaev–Roberts (SR) statistic (for example, see [31] among others). In our case, the classical SR statistic was of the form

$$\text{SRELR} = \sum_{k \in G} \exp \, (-2\text{LLR})_k > C_\alpha, \tag{16}$$

where C_α is the test threshold and is $100(1-\alpha)\%$ percentile of the $\chi^2(1)$ distribution. The set G are integer values representing the moment constraints that will maximize the test statistic. Our proposed test statistic (equation (15)) is developed utilizing approaches introduced by Vexler and Wu [32], and various authors have demonstrated that the SR statistic (see equation (16)) and the CUSUM-type statistic have almost equivalent optimal statistical properties due to their common null-martingale basis [31]. The choice of G is also vital in moment-based test statistics. Fang et al. [25] utilized the probability integral transformation and recommended that, under the test of symmetry, only odd-ordered moments of the transformed data are required in the construction of the test statistic. They further alluded that the use of odd-ordered moments has several benefits that include power against most symmetric alternatives and robustness and performs well under small sample sizes among others.

For our proposed test statistics, we decided to then conduct an extensive Monte Carlo simulation exercise in order to empirically evaluate the suitable choice of G that will give us optimal power under symmetric alternatives in testing for normality. Following the work of Fang et al. [25] as well as Shan et al. [12], we estimated the powers of the test statistics for different alternatives and definitions of odd-ordered moments of G. Table 1 displays a subset of the Monte Caro simulation results. We also considered additional alternatives based on samples of sizes $n = 20$, 50, and 100 at $\alpha = 0.05$. We used size-adjusted critical values for each test statistic, and power for each test was computed using 5,000 replications. The results were that both the CUSUM-type and the Shiryaev–Roberts proposed test statistics with $G = \{3, 5\}$ showed an average power that was greater than that of all other cases under symmetric short-tailed alternatives. In addition, the CUSUM-type test statistics with $G = \{3, 5\}$ showed an average power that was greater than that of all other cases under symmetric long-tailed alternatives. Since our proposed test is meant to perform superior under symmetric alternatives, we

Table 1: Monte Carlo experiments to establish the values of G at $\alpha = 0.05$ for the proposed CUSUM-type (CS) and Shiryaev–Roberts (SR) test statistics.

Distribution	n	$G = \{1, 3\}$		$G = \{3, 5\}$		$G = \{5, 7\}$		$G = \{1, 3, 5\}$		$G = \{3, 5, 7\}$		$G = \{1, 3, 5, 7\}$		$G = \{3, 5, 7, 9\}$	
		CS	SR	CS	SR	CS	SR	CS	SR	CS	SR	CS	SR	CS	SR
Symmetric short-tailed alternative distributions															
Beta (2, 2)	20	0.0486	0.0378	**0.1072**	*0.1116*	0.1020	0.1028	0.0928	0.0820	0.0884	0.1032	0.0838	0.0888	0.0634	0.0886
	50	0.1528	0.1014	**0.2904**	*0.2930*	0.2366	0.2694	0.2536	0.2274	0.2176	0.2918	0.2372	0.2638	0.1822	0.2424
	100	0.3538	0.2480	**0.5676**	0.6062	0.5196	0.5512	0.5430	0.5384	0.5350	*0.6194*	0.5166	0.5688	0.4172	0.4968
Uniform (0, 1)	20	0.1584	0.1056	**0.2998**	*0.3030*	0.2256	0.2402	0.2556	0.2558	0.2134	0.2628	0.2136	0.2404	0.1368	0.2132
	50	0.5214	0.3882	**0.7924**	*0.7874*	0.6768	0.6590	0.7536	0.7482	0.6912	0.7804	0.6984	0.7560	0.5452	0.6714
	100	0.8750	0.8066	**0.9882**	0.9872	0.9596	0.9576	0.9838	0.9806	0.9736	*0.9890*	0.9738	0.9848	0.9532	0.9692
Trunc-norm (−1, 1)	20	0.0898	0.0618	**0.1878**	*0.1920*	0.1512	0.1694	0.1560	0.1534	0.1446	0.1704	0.1404	0.1464	0.1004	0.1386
	50	0.3140	0.2000	**0.5604**	*0.5630*	0.4624	0.4802	0.5144	0.5014	0.4490	0.5612	0.4466	0.4984	0.3384	0.4600
	100	0.6636	0.5412	**0.8866**	*0.9094*	0.8290	0.8398	0.8698	0.8706	0.8570	0.9038	0.8504	0.8866	0.7758	0.8358
Tukey (0, 1, 0.75, 0.75)	20	0.1160	0.0726	**0.2294**	*0.2332*	0.1714	0.1904	0.1878	0.1882	0.1652	0.2118	0.1724	0.1766	0.1062	0.1640
	50	0.3816	0.2640	**0.6448**	*0.6458*	0.5232	0.5442	0.6202	0.6002	0.5330	0.6206	0.5560	0.5880	0.4062	0.5334
	100	0.7598	0.6586	**0.9428**	0.9460	0.9014	0.8948	0.9362	0.9248	0.9132	*0.9550*	0.9228	0.9382	0.8568	0.8984
Symmetric long-tailed alternative distributions															
T (4)	20	**0.2396**	*0.2530*	0.2208	0.2060	0.1828	0.1782	0.2286	0.2342	0.1990	0.2042	0.1968	0.2254	0.1798	0.1850
	50	0.4484	0.4212	**0.4770**	0.4600	0.4084	0.4220	0.4426	*0.4810*	0.4440	0.4596	0.4374	0.4684	0.4142	0.4302
	100	0.6630	0.6520	**0.7264**	0.7094	0.6896	0.6950	0.7062	0.7146	0.7018	0.7142	0.7012	*0.7226*	0.6556	0.6650
Logistic (0, 1)	20	**0.1192**	*0.1196*	0.1076	0.1014	0.0902	0.0846	0.1050	0.1178	0.1038	0.0968	0.1002	0.1080	0.0888	0.0908
	50	0.1840	0.1794	**0.1990**	0.1940	0.1708	0.1590	0.1890	0.1904	0.1736	0.1884	0.1790	*0.1976*	0.1546	0.1640
	100	0.2796	0.2490	**0.3082**	0.3216	0.2716	0.2766	0.3070	*0.3254*	0.2976	0.3104	0.2902	0.3204	0.2448	0.2712
Double-exp (0, 1)	20	0.2396	*0.2708*	**0.2598**	0.2270	0.1772	0.1792	0.2364	0.2660	0.2116	0.2088	0.2196	0.2382	0.1732	0.1738
	50	0.5032	0.4770	**0.5446**	0.5252	0.4410	0.4422	0.5322	*0.5450*	0.4926	0.4870	0.4980	0.5222	0.4412	0.4524
	100	0.7626	0.7334	**0.8102**	0.8158	0.7412	0.7470	0.8092	*0.8214*	0.7918	0.8058	0.7846	0.8130	0.7360	0.7328
Cauchy (0, 1)	20	**0.8588**	0.8554	0.8408	0.8318	0.7742	0.7580	0.8464	*0.8678*	0.8138	0.8140	0.8226	0.8420	0.7820	0.7744
	50	0.9940	0.9950	**0.9958**	*0.9970*	0.9944	0.9906	0.9956	0.9956	0.9946	0.9968	0.9938	0.9952	0.9936	0.9938
Asymmetric alternative distributions															
Gamma (2, 1)	50	**0.8614**	*0.8306*	0.5636	0.4900	0.2820	0.2240	0.8172	0.7782	0.5282	0.4230	0.7794	0.7254	0.5682	0.4202
	100	**0.9950**	*0.9910*	0.8266	0.7686	0.4182	0.3356	0.9890	0.9818	0.8372	0.7216	0.9854	0.9736	0.8536	0.7208
Weibull (2, 1)	50	**0.2794**	*0.2892*	0.1680	0.1302	0.1354	0.1128	0.2568	0.2340	0.1590	0.1338	0.2358	0.2160	0.2070	0.1556
	100	**0.5804**	*0.5506*	0.2294	0.1870	0.1786	0.1298	0.5048	0.4496	0.2750	0.1950	0.4662	0.4062	0.3260	0.2106
SN (0, 1, 5)	50	**0.5268**	*0.5064*	0.2674	0.2010	0.1688	0.1428	0.4772	0.4332	0.2750	0.2040	0.4262	0.3864	0.3256	0.2254
	100	**0.8706**	*0.8402*	0.4290	0.3392	0.2192	0.1646	0.8430	0.7732	0.4754	0.3166	0.8062	0.7440	0.5418	0.3506

Bold represents the most superior CS-type statistic and italicized represents the most superior SR statistic.

recommend to use $G = \{3, 5\}$ since the proposed test statistics with $G = \{3, 5\}$ are simple and provide relatively high levels of power under symmetric alternatives. The results of our Monte Carlo simulation experiments are consistent with that of Fang et al. [25]. In this article, we denoted CSELR for the CUSUM-type test statistic and SRELR for the Shiryaev–Roberts proposed test statistic. A schematic algorithm of the testing procedure is shown in Figure 1.

3. Monte Carlo Simulation Procedures

We utilized the R statistical package for all the simulation procedures. Firstly, size-adjusted critical values for the proposed test statistics were determined. In order to achieve this, we used 50,000 replications, and without loss of generality, data were simulated from a standard normal distribution at stipulated sample sizes and α-levels. Only samples of sizes 20 to 100 were considered (see Table 2). This was entirely motivated by the need to utilize samples that commonly arise in practice.

For power comparisons, twelve selected competitor tests were considered. The choice of these tests was guided by potential competitor tests, thus tests developed using similar characterization techniques as well as well-known powerful classical normality tests. Three broad categories of these competitor tests were established. These include EDF-based tests, moment-based tests, and the Shapiro–Wilk-based tests. For the EDF-based tests, we opted for the Anderson–Darling (AD) test [27, 28], the Lilliefors (LL) test which is well known as the modified Kolmogorov–Smirnov test [29], the Cramér–von Mises (CVM) test [30], and the H_n test [33]. Moment-based tests included the Jarque–Bera's (JB) test [20, 21], the robust Jarque–Bera's (JB) test [20, 21], the kurtosis (b_2) test [14], the skewness ($\sqrt{b_1}$) test [14], the D'Agostino–Pearson k^2 (DP) test [22], and the simple and exact empirical likelihood ratio (SEELR) test based on moment relations [12]. Lastly, the other categories of competitor tests consisted of the classical well-known and powerful Shapiro–Wilk (SW) test [13] and the Shapiro–Francia's (SF) test [15] which is a modification of the SW test. Most of these competitor tests have proved to be powerful against a wide range of alternatives including symmetric ones [33–39].

In terms of the alternative distributions, we considered distributions that cover a wide range of symmetric alternative distributional properties. Following Esteban et al. [40] and Torabi et al. [33], we considered the following alternative distributions, which can be classified into two broad sets of symmetric alternative distributions: (1) symmetric short-tailed distributions and (2) symmetric long-tailed distributions. In order to evaluate our proposed tests under asymmetric alternatives, we opted for a third set (3) of asymmetric alternatives:

(1) Set 1: symmetric short-tailed distributions:
   - (i) The beta distribution with parameters (3, 3), (2, 2), (1, 1), and (0.5, 0.5)
   - (ii) The uniform distribution, $U(a, b)$ with $a = 0$ and $b = 1$
   - (iii) The logit-normal distribution with $\mu = 0$ and $\sigma = 1$
   - (iv) The truncated standard normal distribution at a and b, i.e., $(-2, 2)$ and $(-1, 1)$
   - (v) Tukey's lambda distribution with $\lambda = 0.25, 0.75$, and 1.25

(2) Set 2: symmetric long-tailed distributions:
   - (i) Student's t distribution with 2, 4, 7, and 15 degrees of freedom.
   - (ii) The Cauchy distribution with $x_0 = 0$ and $\gamma = 1$
   - (iii) The logistic distribution with parameters $\mu = 0$ (location) and $\sigma = 1$ (scale)
   - (iv) The double exponential distribution (also known as the Laplace distribution) with parameters μ (location) and λ (scale)
   - (v) Tukey's lambda distribution with $\lambda = -0.10, -0.15$, and -0.25

(3) Set 3: asymmetric distributions:
   - (i) The gamma distribution with parameters (2, 1)
   - (ii) The Weibull distribution with parameters (2, 1)
   - (iii) The skewed normal distribution with parameters (0, 1, 5)
   - (iv) The skewed Cauchy distribution with parameters (0, 2, 5)
   - (v) The beta distribution with parameters (2, 1) and (3, 1.5)

For power simulation, 10,000 samples each of size $n = 20, 30, 50, 80$, and 100 were obtained under the various alternative distributions. Power was computed by considering the number of times the test rejected the null hypothesis over the total number of replications. A numerical bootstrap study on real data was conducted to assess the robustness and applicability of the proposed tests. However, it was necessary to first assess the type I error control of our proposed tests before the power study.

3.1. Type I Error Control. Here, we provide the values of type I error rates along with the associated standard errors of the proposed tests for $\alpha = 0.01, 0.05$, and 0.10. In order to compute these quantities, for each nominal alpha, we generated 500,000 random samples from a standard normal distribution, each corresponding to sample size $n = 20, 30, 50, 80$, and 100. The results presented in Table 3 show that the proposed tests control type I error very well. Figure 2 includes plots of the simulated type error rates only for $\alpha = 0.05$ for all the sample sizes considered. The plots for the empirical cumulative probability function of the simulated p values for $n = 20$, $\alpha = 0.01$, and $\alpha = 0.10$ were omitted since their plots were more or less the same as those for other sample sizes and $\alpha = 0.05$, respectively. It is evident that the plots produced the expected appearance in all the simulated scenarios. That is, the plots show close to the α-level of simulated type I error rates. The closeness of the estimated probabilities of type I error to the nominal value ($\alpha = 0.05$) attests that the GoF test does perform as expected. These results were extended in order to evaluate the type I

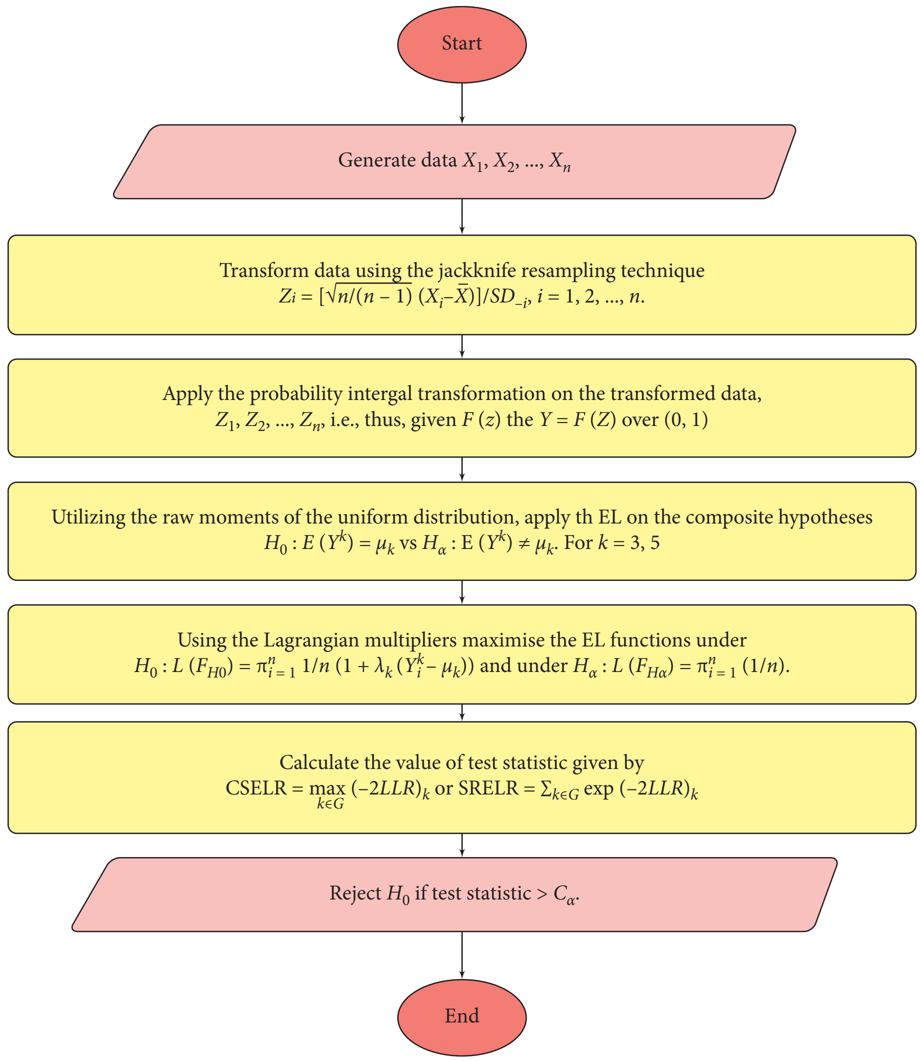

Figure 1: Schematic algorithm of the testing procedure.

Table 2: Size-adjusted critical values for the proposed tests at $\alpha = 0.01 - 0.10$.

Size-adjusted critical values for CUSUM-type-based test statistic at (α)-H_0: $X \sim N(0,1)$										
n	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
20	0.4206	0.3437	0.3004	0.2742	0.2547	0.2378	0.2249	0.2140	0.2044	0.1956
30	0.4328	0.3535	0.3112	0.2833	0.2625	0.2466	0.2321	0.2198	0.2095	0.2009
50	0.4458	0.3635	0.3226	0.2920	0.2685	0.2509	0.2361	0.2232	0.2124	0.2022
80	0.4562	0.3695	0.3255	0.2945	0.2720	0.2544	0.2388	0.2260	0.2148	0.2043
100	0.4581	0.3816	0.3348	0.3036	0.2809	0.2627	0.2477	0.2335	0.2219	0.2108
Size-adjusted critical values for Shiryaev–Roberts test statistic at (α)-H_0: $X \sim N(0,1)$										
n	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
20	2.7840	2.5992	2.5136	2.4619	2.4231	2.3926	2.3676	2.3465	2.3280	2.3115
30	2.8129	2.6242	2.5361	2.4790	2.4384	2.4067	2.3813	2.3586	2.3384	2.3207
50	2.8519	2.6658	2.5725	2.5066	2.4595	2.4205	2.3908	2.3672	2.3476	2.3280
80	2.8821	2.6758	2.5823	2.5177	2.4648	2.4323	2.4031	2.3761	2.3514	2.3311
100	2.8846	2.6920	2.5877	2.5190	2.4678	2.4368	2.3962	2.3792	2.3559	2.3371

error control when simulating from a normal distribution with varying parameters of μ and σ^2. We considered various scenarios which include $N(0, 5^2)$, $N(5, 5^2)$, $N(7, 15^2)$, $N(15, 25^2)$, and $N(50, 75^2)$ for samples of sizes 20, 50, 100 at alpha levels of 0.01, 0.05, and 0.10 (see Table 4). Similarly, as observed in Table 3, the estimated probabilities of type I error were close to the respective nominal values which shows that the GoF test does perform as expected. It is important to note that various alternative methods that can also be used to assess the closeness of the simulated type I error rates to the nominal size alpha are available in the literature. The most popular one is based on the central limit

Table 3: Type I error rates along with the standard error for the proposed test statistics using $N(0,1)$.

Alpha	Simulated probabilities of the type I error along with standard error		
	Sample size	CUSUM-type test statistic	Shiryaev–Roberts test statistic
0.01	20	$0.009214 \pm 1.3509e-04$	$0.009346 \pm 1.3606e-04$
	30	$0.010163 \pm 1.4181e-04$	$0.010291 \pm 1.4269e-04$
	50	$0.010589 \pm 1.4469e-04$	$0.009849 \pm 1.3963e-04$
	80	$0.010893 \pm 1.4675e-04$	$0.009613 \pm 1.3796e-04$
	100	$0.009706 \pm 1.3862e-04$	$0.009916 \pm 1.4003e-04$
0.05	20	$0.050133 \pm 3.0859e-04$	$0.049785 \pm 3.0758e-04$
	30	$0.049841 \pm 3.0774e-04$	$0.050578 \pm 3.0989e-04$
	50	$0.052238 \pm 3.1464e-04$	$0.049027 \pm 3.0534e-04$
	80	$0.051434 \pm 3.1235e-04$	$0.050018 \pm 3.0826e-04$
	100	$0.048295 \pm 3.0317e-04$	$0.051756 \pm 3.1327e-04$
0.10	20	$0.102066 \pm 4.2812e-04$	$0.100521 \pm 4.2523e-04$
	30	$0.100938 \pm 4.2601e-04$	$0.098578 \pm 4.2156e-04$
	50	$0.101902 \pm 4.2781e-04$	$0.099039 \pm 4.2242e-04$
	80	$0.101586 \pm 4.2722e-04$	$0.099829 \pm 4.2392e-04$
	100	$0.095552 \pm 4.1573e-04$	$0.101395 \pm 4.2686e-04$

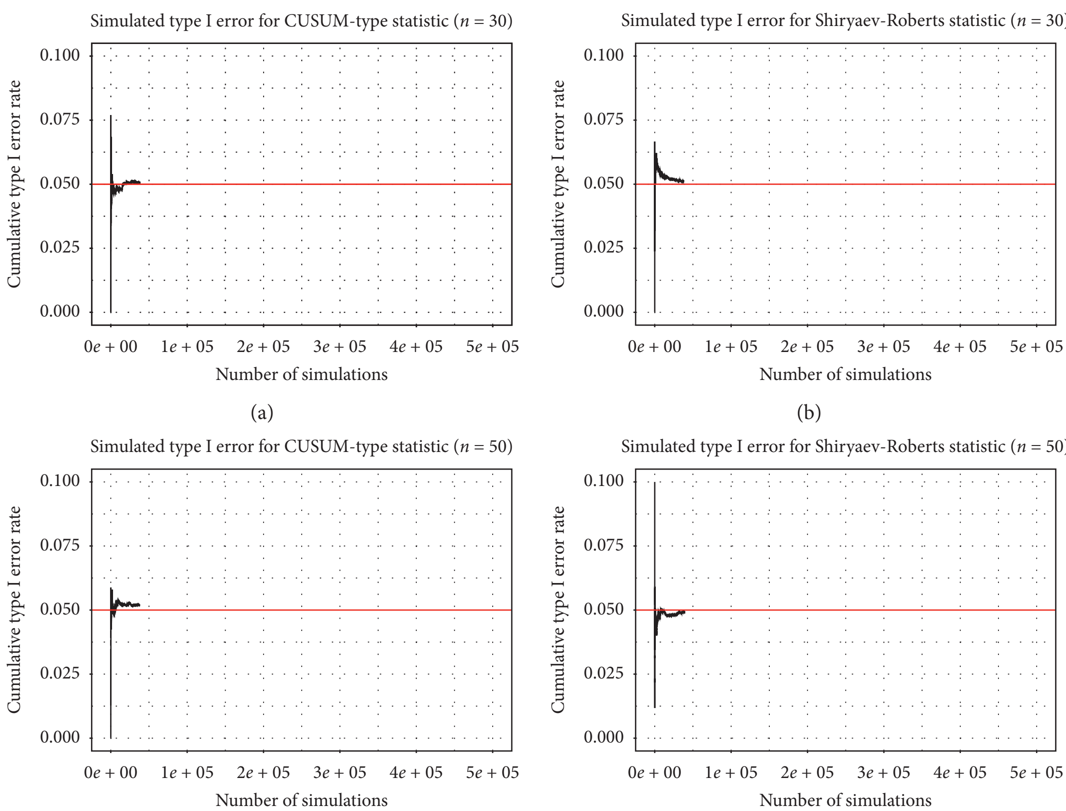

Figure 2: Continued.

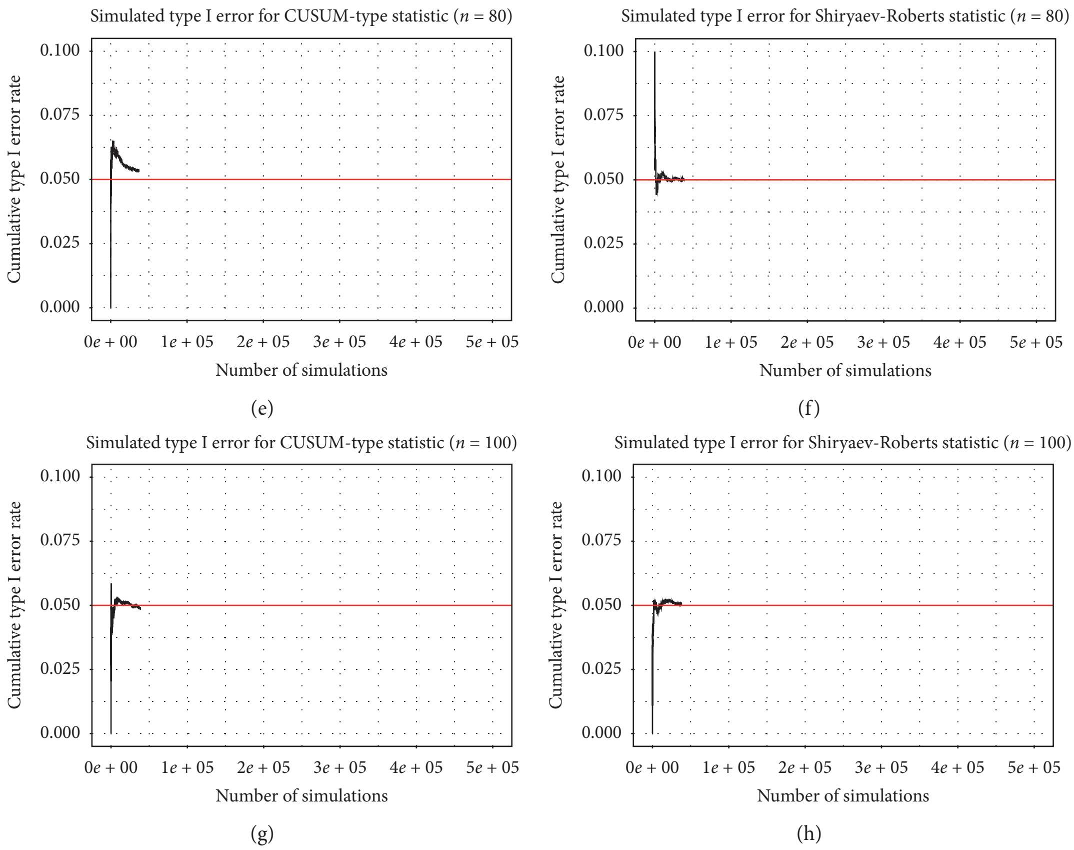

Figure 2: Cumulative type I error rates for $\alpha = 0.05$ at different sample sizes ($n = 30, 50, 80$, and 100) using 500,000 simulations.

theorem and it was described in detail by Batsidis et al. [41]. Once the type I error rates were examined, we then proceeded to evaluate the powers of the proposed tests to determine how well they would detect departures from normality and to see their power performance compared to those of the selected competing tests.

3.2. Monte Carlo Power Simulation Results. Results for the Monte Carlo power comparisons are presented. Bold numbers in all tables represent the two most superior tests under the respective simulated scenarios. From Tables 5 and 6, we found out that when the alternative distributions are short-tailed and symmetric, our proposed tests performed quite well. Under these symmetric alternative cases, our proposed tests (SRELR and CSELR) significantly outperformed all other studied tests. Tests based on LL, JB, RJB, and $\sqrt{b_1}$ have the least power as compared to other tests. In general, the tests based on SRELR, CSELR, DP, SW, and b_2 are the most powerful under these symmetric short-tailed alternative distributions.

For symmetric long-tailed alternatives (see Tables 7 and 8), the tests based on RJB, b_2, SF, and JB are more superior, whereas the tests based on $\sqrt{b_1}$, LL, and SEELR are the least powerful. Our proposed tests performed slightly lesser than the DP test but were comparable to the SW test. It is important to note that, in all of the cases under these symmetric long-tailed alternatives, our proposed tests outperformed all the EDF-based tests.

For the considered asymmetric alternatives (see Table 9), the tests based on SEELR, SW, SF, and AD are more superior, whereas the tests based on b_2, RJB, and SRELR are the least powerful. Our proposed test based on CSELR performed slightly lesser than the JB test but was comparable to the LL test. It is important to note that, in all of the cases under these asymmetric alternatives, our proposed test based on CSELR outperformed the SRELR-based test.

In order to get a clearer visualisation of the performance of the different normality tests, the ranking procedure was used. Tables 10 to 12 contain the ranking of all the tests considered in this study according to the average powers computed from the values in Tables 5–8 and 9, respectively. The rank of power is based on the set of alternative distributions and sample sizes, respectively. Using average powers, we can select the tests that are, on average, most powerful against the alternatives from the given sets of alternatives. It should be noted that, under all the symmetric simulated scenarios, our proposed tests (SRELR and CSELR) were comparable in power.

From Table 10, it can be clearly seen that our proposed tests (SRELR and CSELR) are the most powerful tests for both small and moderate sample sizes under symmetric short-tailed alternatives. This is followed rather closely by the DP test. The results of the total rank based on all sample sizes (i.e., $n = 20$ to 100) show that our proposed tests (SRELR and CSELR) are overly the most superior tests for symmetric short-tailed distributions.

Table 4: Type I error rates along with the standard error for the proposed test statistics using $N(\mu, \sigma^2)$.

$N(\mu, \sigma^2)$	Simulated probabilities of the type I error along with standard error			
	Alpha	Sample size	CUSUM-type test statistic	Shiryaev–Roberts test statistic
		20	0.009434 ± 1.3665$e-04$	0.009415 ± 1.3655$e-04$
	0.01	50	0.010655 ± 1.4515$e-04$	0.009971 ± 1.4047$e-04$
		100	0.009746 ± 1.3890$e-04$	0.009969 ± 1.4046$e-04$
		20	0.049903 ± 3.0701$e-04$	0.049897 ± 3.0788$e-04$
$N(0, 5^2)$	0.05	50	0.051580 ± 3.1278$e-04$	0.049959 ± 3.0808$e-04$
		100	0.048599 ± 3.0406$e-04$	0.051496 ± 3.1254$e-04$
		20	0.102288 ± 4.2854$e-04$	0.099721 ± 4.2373$e-04$
	0.10	50	0.102146 ± 4.2825$e-04$	0.099881 ± 4.2418$e-004$
		100	0.096571 ± 4.1769$e-04$	0.102983 ± 4.2982$e-04$
		20	0.009358 ± 1.3612$e-04$	0.009790 ± 1.3921$e-04$
	0.01	50	0.010421 ± 1.4357$e-04$	0.009792 ± 1.3920$e-04$
		100	0.009779 ± 1.3912$e-04$	0.009636 ± 1.3812$e-04$
		20	0.050542 ± 3.0979$e-04$	0.049276 ± 3.0608$e-04$
$N(5, 5^2)$	0.05	50	0.050920 ± 3.1087$e-04$	0.048651 ± 3.0424$e-04$
		100	0.048685 ± 3.0434$e-04$	0.051655 ± 3.1296$e-04$
		20	0.102156 ± 4.2829$e-04$	0.099039 ± 4.2243$e-04$
	0.10	50	0.103031 ± 4.2990$e-04$	0.098468 ± 4.2135$e-04$
		100	0.095373 ± 4.1537$e-04$	0.101721 ± 4.2747$e-04$
		20	0.009227 ± 1.3520$e-04$	0.009651 ± 1.3824$e-04$
	0.01	50	0.010575 ± 1.4462$e-04$	0.010104 ± 1.4138$e-04$
		100	0.009551 ± 1.3751$e-04$	0.009495 ± 1.3713$e-04$
		20	0.050768 ± 3.1043$e-04$	0.048988 ± 3.0522$e-04$
$N(7, 15^2)$	0.05	50	0.050882 ± 3.1077$e-04$	0.048730 ± 3.0448$e-04$
		100	0.049044 ± 3.0540$e-04$	0.050758 ± 3.1041$e-04$
		20	0.101926 ± 4.2786$e-04$	0.100764 ± 4.2569$e-04$
	0.10	50	0.102209 ± 4.2838$e-04$	0.099809 ± 4.2387$e-04$
		100	0.096280 ± 4.1714$e-04$	0.101345 ± 4.2678$e-04$
		20	0.009416 ± 1.3648$e-04$	0.009651 ± 1.3824$e-04$
	0.01	50	0.010531 ± 1.4432$e-04$	0.009910 ± 1.4007$e-04$
		100	0.010028 ± 1.4086$e-04$	0.009329 ± 1.3592$e-04$
		20	0.049686 ± 3.0729$e-04$	0.049465 ± 3.0664$e-04$
$N(15, 25^2)$	0.05	50	0.051571 ± 3.1275$e-04$	0.049887 ± 3.0788$e-04$
		100	0.048452 ± 3.0363$e-04$	0.050863 ± 3.1072$e-04$
		20	0.101373 ± 4.2683$e-04$	0.099383 ± 4.2307$e-04$
	0.10	50	0.101062 ± 4.2625$e-04$	0.099168 ± 4.2267$e-04$
		100	0.095026 ± 4.1470$e-04$	0.102562 ± 4.2904$e-04$
		20	0.009383 ± 1.3621$e-04$	0.009415 ± 1.3639$e-04$
	0.01	50	0.010662 ± 1.4521$e-04$	0.010074 ± 1.4119$e-04$
		100	0.009592 ± 1.3782$e-04$	0.009724 ± 1.3874$e-04$
		20	0.049308 ± 3.0618$e-04$	0.049587 ± 3.0699$e-04$
$N(50, 75^2)$	0.05	50	0.051753 ± 3.1325$e-04$	0.049020 ± 3.0533$e-04$
		100	0.048467 ± 3.0368$e-04$	0.050790 ± 3.1050$e-04$
		20	0.101744 ± 4.2752$e-04$	0.099688 ± 4.2366$e-04$
	0.10	50	0.100745 ± 4.2563$e-04$	0.099476 ± 4.2326$e-04$
		100	0.096317 ± 4.1722$e-04$	0.102060 ± 4.2809$e-04$

Note. Monte Carlo simulations were conducted using 500,000 replications.

For symmetric long-tailed alternatives (see Table 11), generally the RJB test was the most powerful in both small and moderate sample sizes. Our proposed tests had comparable power with the AD test under small samples for symmetric long-tailed alternatives. However, under moderate sample sizes, our proposed tests were slightly more powerful than the DP and SW tests. Lastly, considering all the sample sizes under symmetric long-tailed alternatives, our proposed tests were comparable to the SW test.

Lastly, under asymmetric alternatives (see Table 12), our proposed test based on CSELR performed better than the SRELR-based test. It is also important to note that our related test, the SEELR, outperformed all other tests under these considered asymmetric alternatives. It is also important to note that, unlike some of the competitor tests, our proposed tests were consistent in power under all alternative distributions for all simulated scenarios.

4. Real Data Study

We used the snowfall dataset to examine the applicability of the proposed test on real data. The snowfall dataset consists of 63 snow precipitation values that were recorded from the

TABLE 5: Monte Carlo power comparisons from short-tailed symmetric alternative distributions at $\alpha = 0.05$.

Symmetric short-tailed alternative distributions: set 1 (1 of 2)															
Distribution	n	Selected EDF-based tests				Selected moment-based tests						SW tests		Proposed tests	
		AD	LL	CVM	H_n	JB	RJB	b_2	$\sqrt{b_1}$	DP	SEELR	SW	SF	CSELR	SRELR
	20	0.0426	0.0433	0.0431	0.0490	0.0098	0.0117	0.0389	0.0117	0.0244	0.0471	0.0379	0.0211	**0.0708**	**0.0702**
	30	0.0512	0.0495	0.0497	0.0588	0.0036	0.0057	0.0357	0.0084	0.0436	0.0396	0.0460	0.0180	**0.0873**	**0.0889**
Beta (3, 3)	50	0.0754	0.0580	0.0670	0.0700	0.0015	0.0021	0.0635	0.0044	0.0983	0.0332	0.0694	0.0236	**0.1387**	**0.1404**
	80	0.1008	0.0638	0.0943	0.0970	0.0021	0.0005	0.1268	0.0027	0.1786	0.0304	0.1086	0.0413	**0.2315**	**0.2303**
	100	0.1278	0.0860	0.1121	0.1270	0.0086	0.0004	0.2080	0.0025	0.2437	0.0372	0.1556	0.0605	**0.2783**	**0.3027**
	20	0.0567	0.0499	0.0566	0.0629	0.0048	0.0055	0.0621	0.0081	0.0363	0.0714	0.0516	0.0200	**0.1065**	**0.1086**
	30	0.0733	0.0614	0.0691	0.0828	0.0012	0.0027	0.0854	0.0044	0.0872	0.0570	0.0773	0.0304	**0.1605**	**0.1638**
Beta (2, 2)	50	0.1328	0.0816	0.1142	0.1152	0.0014	0.0005	0.1713	0.0020	0.2116	0.0678	0.1521	0.0533	**0.2829**	**0.2837**
	80	0.2280	0.1252	0.1863	0.2016	0.0100	0.0001	0.3941	0.0013	0.3867	0.1097	0.3140	0.1382	**0.4819**	**0.4840**
	100	0.3072	0.1493	0.2380	0.2668	0.0487	0.0001	0.5433	0.0024	0.4871	0.1658	0.4526	0.2130	**0.5835**	**0.6075**
	20	0.1622	0.0959	0.1419	0.1527	0.0030	0.0038	0.2158	0.0050	0.1550	0.2570	0.1991	0.0843	**0.3025**	**0.3126**
	30	0.3071	0.1459	0.2300	0.2446	0.0013	0.0013	0.3659	0.0040	0.3914	0.3267	0.3814	0.1757	**0.4945**	**0.5121**
Beta (1, 1)	50	0.5760	0.2531	0.4399	0.4689	0.0130	0.0003	0.7059	0.0015	**0.8021**	0.5623	0.7468	0.4719	**0.7826**	0.7780
	80	0.8656	0.4558	0.7236	0.7518	0.3660	0.0000	0.9538	0.0012	**0.9808**	0.8727	**0.9735**	0.8670	0.9563	0.9562
	100	0.9523	0.5850	0.8343	0.8465	0.7299	0.0398	0.9911	0.0014	**0.9965**	0.9573	**0.9954**	0.9642	0.9849	0.9851
	20	0.6144	0.3070	0.5015	0.5145	0.0057	0.0050	0.5753	0.0140	0.4586	**0.7233**	**0.7220**	0.5020	0.6994	0.6900
	30	0.8551	0.4838	0.7348	0.7511	0.0052	0.0025	0.8098	0.0082	0.7650	0.8850	**0.9380**	0.8090	**0.9094**	0.9031
Beta (0.5, 0.5)	50	0.9909	0.7972	0.9562	0.9587	0.3781	0.0006	0.9816	0.0060	0.9014	0.9914	**0.9998**	0.9914	**0.9937**	0.9919
	80	0.9999	0.9708	0.9987	0.9999	0.9867	0.1121	0.9996	0.0036	0.9612	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
	100	**1.0000**	0.9946	0.9998	**1.0000**	0.9998	0.8388	0.9999	0.0045	0.9878	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
	20	0.1677	0.0969	0.1431	0.1450	0.0028	0.0033	0.2175	0.0063	0.1400	0.2752	0.1960	0.0825	**0.3025**	**0.3092**
	30	0.3056	0.1343	0.2212	0.2390	0.0012	0.0013	0.3838	0.0033	0.3880	0.3258	0.3720	0.1647	**0.5055**	**0.5072**
Uniform (0, 1)	50	0.5742	0.2555	0.4420	0.4604	0.0106	0.0000	0.7005	0.0022	**0.7983**	0.5650	0.7568	0.4813	**0.7886**	0.7859
	80	0.8642	0.4495	0.7204	0.7447	0.3731	0.0001	0.9500	0.0020	**0.9789**	0.8782	**0.9744**	0.8651	0.9587	0.9549
	100	0.9438	0.5876	0.8484	0.8470	0.7238	0.0417	0.9907	0.0031	**0.9970**	0.9562	**0.9958**	0.9651	0.9865	0.9856
	20	0.0619	0.0481	0.0586	0.0549	0.0056	0.0060	0.0640	0.0080	0.0396	0.0732	0.0533	0.0242	**0.1094**	**0.1132**
	30	0.0870	0.0623	0.0762	0.0788	0.0020	0.0024	0.0898	0.0044	0.0886	0.0640	0.0848	0.0302	**0.1658**	**0.1704**
Logit-norm (0, 1)	50	0.1446	0.0788	0.1244	0.1248	0.0040	0.0003	0.1745	0.0013	0.2449	0.0664	0.1616	0.0529	**0.2935**	**0.2998**
	80	0.2500	0.1342	0.1983	0.2026	0.0132	0.0002	0.3945	0.0016	**0.5049**	0.1003	0.3255	0.1437	**0.5025**	0.4985
	100	0.3303	0.1598	0.2635	0.2802	0.0518	0.0000	0.5557	0.0022	**0.6622**	0.1568	0.4702	0.2245	0.6035	**0.6246**

Table 6: Monte Carlo power comparisons from short-tailed symmetric alternative distributions at $\alpha = 0.05$.

Distribution	n	Symmetric short-tailed alternative distributions: set 1 (2 of 2)													
		Selected EDF-based tests				Selected moment-based tests						SW tests		Proposed tests	
		AD	LL	CVM	H_n	JB	RJB	b_2	$\sqrt{b_1}$	DP	SEELR	SW	SF	CSELR	SRELR
	20	0.0405	0.0375	0.0392	0.0426	0.0090	0.0119	0.0278	0.0120	0.0187	0.0415	0.0340	0.0227	**0.0563**	**0.0577**
	30	0.0429	0.0426	0.0455	0.0466	0.0036	0.0048	0.0287	0.0067	0.0280	0.0318	0.0363	0.0179	**0.0648**	**0.0685**
Trunc-norm (−2, 2)	50	0.0517	0.0452	0.0463	0.0489	0.0010	0.0014	0.0418	0.0043	0.0651	0.0265	0.0539	0.0187	**0.1032**	**0.0972**
	80	0.0743	0.0583	0.0697	0.0734	0.0011	0.0010	0.0949	0.0031	0.1462	0.0231	0.0895	0.0303	**0.1595**	**0.1639**
	100	0.0913	0.0628	0.0756	0.0820	0.0027	0.0004	0.1508	0.0025	**0.2001**	0.0276	0.1239	0.0343	0.1977	**0.2196**
	20	0.0982	0.0612	0.0842	0.0882	0.0025	0.0039	0.1240	0.0044	0.0713	0.1387	0.1033	0.0389	**0.1801**	**0.1844**
	30	0.1556	0.0903	0.1188	0.1453	0.0006	0.0018	0.1922	0.0031	0.2090	0.1596	0.1935	0.0734	**0.3065**	**0.3221**
Trunc-norm (−1, 1)	50	0.3105	0.1322	0.2245	0.2304	0.0027	0.0004	0.4480	0.0020	0.5439	0.2718	0.4479	0.1987	**0.5454**	**0.5491**
	80	0.5663	0.2352	0.4092	0.4562	0.1024	0.0000	0.7887	0.0014	**0.8729**	0.5412	0.8002	0.5372	0.8042	**0.8133**
	100	0.7103	0.3131	0.5263	0.5481	0.3173	0.0027	0.9130	0.0008	**0.9508**	0.7245	**0.9274**	0.7452	0.8963	0.9075
	20	0.0401	0.0433	0.0452	0.0432	0.0176	0.0237	0.0305	0.0206	0.0274	0.0412	0.0373	0.0269	**0.0507**	**0.0491**
	30	0.0418	0.0420	0.0443	0.0497	0.0129	0.0149	0.0235	0.0163	0.0285	0.0320	0.0385	0.0219	**0.0596**	**0.0586**
Tukey (0, 1, 0.25, 0.25)	50	0.0512	0.0483	0.0552	0.0532	0.0041	0.0070	0.0248	0.0121	0.0447	0.0314	0.0408	0.0195	**0.0749**	**0.0756**
	80	0.0603	0.0504	0.0626	0.0581	0.0031	0.0031	0.0455	0.0090	0.0776	0.0254	0.0530	0.0194	**0.1053**	**0.1003**
	100	0.0664	0.0575	0.0636	0.0671	0.0042	0.0026	0.0628	0.0078	0.0952	0.0291	0.0572	0.0217	**0.1159**	**0.1321**
	20	0.1236	0.0743	0.1056	0.1048	0.0033	0.0031	0.1564	0.0061	0.1017	0.1765	0.1356	0.0548	**0.2246**	**0.2306**
	30	0.1977	0.0969	0.1578	0.1672	0.0005	0.0021	0.2546	0.0030	0.2763	0.1984	0.2421	0.0986	**0.3744**	**0.3751**
Tukey (0, 1, 0.75, 0.75)	50	0.4028	0.1812	0.3040	0.3216	0.0042	0.0002	0.5498	0.0020	**0.6529**	0.3474	0.5383	0.2679	**0.6506**	0.6465
	80	0.6996	0.3116	0.5387	0.5649	0.1758	0.0000	0.8664	0.0010	**0.9303**	0.6599	0.8745	0.6524	**0.8844**	0.8821
	100	0.8181	0.4289	0.6699	0.6751	0.4598	0.0086	0.9418	0.0010	**0.9822**	0.8220	0.9421	0.8391	0.9467	**0.9486**
	20	0.2083	0.1096	0.1617	0.1810	0.0.0032	0.0042	0.2589	0.0061	0.1767	0.2997	0.2426	0.1083	**0.3514**	**0.3530**
	30	0.3515	0.1575	0.2697	0.2763	0.0012	0.0014	0.4255	0.0040	0.4339	0.4012	0.4534	0.2240	**0.5569**	**0.5689**
Tukey (0, 1, 1.25, 1.25)	50	0.6608	0.3045	0.5147	0.5101	0.0205	0.0002	0.7786	0.0017	**0.8414**	0.6650	0.8221	0.5744	**0.8466**	0.8374
	80	0.9181	0.5348	0.7963	0.8105	0.4832	0.0004	0.9725	0.0011	**0.9878**	0.9283	0.9768	0.9301	0.9773	**0.9797**
	100	0.9752	0.6676	0.9032	0.9114	0.8247	0.0766	0.9946	0.0010	**0.9982**	0.9824	0.9942	0.9900	0.9951	**0.9954**

Table 7: Monte Carlo power comparisons from long-tailed symmetric alternative distributions at $\alpha = 0.05$.

Symmetric long-tailed alternative distributions: set 2 (1 of 2)															
Distribution	n	Selected EDF-based tests				Selected moment-based tests						SW tests		Proposed tests	
		AD	LL	CVM	H_n	JB	RJB	b_2	$\sqrt{b_1}$	DP	SEELR	SW	SF	CSELR	SRELR
	20	0.5124	0.4439	0.5058	0.4895	0.5678	**0.6325**	0.5421	0.4978	0.5644	0.3766	0.5209	**0.5825**	0.5055	0.4940
	30	0.6801	0.5987	0.6605	0.6356	0.7220	**0.7737**	0.7323	0.5987	0.7034	0.4316	0.6744	**0.7361**	0.6757	0.6672
t (2)	50	0.8499	0.7722	0.8371	0.8387	0.8814	**0.9198**	**0.8984**	0.7006	0.8584	0.4838	0.8637	0.8891	0.8685	0.8621
	80	0.9588	0.9172	0.9519	0.9455	0.9716	**0.9799**	**0.9725**	0.7838	0.9584	0.5336	0.9626	0.9715	0.9667	0.9669
	100	0.9847	0.9593	0.9791	0.9812	0.9894	**0.9930**	0.9902	0.8123	0.9809	0.5891	0.9851	**0.9925**	0.9872	0.9877
	20	0.2143	0.1798	0.2053	0.1949	**0.2885**	**0.3270**	0.2794	0.2569	0.2864	0.1737	0.2431	0.2790	0.2225	0.2137
	30	0.3010	0.2200	0.2800	0.2583	0.3832	**0.4341**	**0.3933**	0.3180	0.3751	0.2090	0.3172	0.3857	0.3173	0.3086
t (4)	50	0.4185	0.3068	0.3903	0.3602	0.5357	**0.5864**	**0.5442**	0.3914	0.5078	0.2346	0.4556	0.5385	0.4639	0.4544
	80	0.5567	0.4182	0.5193	0.5030	0.6957	**0.7466**	**0.7248**	0.4663	0.6530	0.2686	0.6330	0.6862	0.6335	0.6319
	100	0.6513	0.4940	0.6015	0.6104	0.7728	**0.8137**	**0.7824**	0.5013	0.7337	0.2824	0.7072	0.7633	0.7091	0.7196
	20	0.1194	0.0899	0.1110	0.1058	0.1640	**0.1937**	0.1507	0.1474	**0.1716**	0.1014	0.1360	0.1612	0.1145	0.1178
	30	0.1512	0.1002	0.1298	0.1272	0.2150	**0.2516**	0.2093	0.1808	**0.2109**	0.1164	0.1664	0.2091	0.1511	0.1485
t (7)	50	0.1884	0.1260	0.1642	0.1570	**0.3040**	**0.3390**	0.2945	0.2208	0.2847	0.1390	0.2370	0.2889	0.2243	0.2153
	80	0.2311	0.1622	0.2090	0.2172	0.4040	**0.4387**	**0.4144**	0.2640	0.3543	0.1478	0.3105	0.3938	0.3095	0.3109
	100	0.2684	0.1797	0.2524	0.2380	0.4543	**0.4928**	**0.4802**	0.2701	0.4140	0.1648	0.3696	0.4347	0.3623	0.3676
	20	0.0703	0.0625	0.0679	0.0640	**0.0951**	**0.1166**	0.0888	0.0876	0.0914	0.0632	0.0770	0.0912	0.0674	0.0686
	30	0.0767	0.0660	0.0708	0.0634	0.1087	**0.1280**	**0.1168**	0.0961	0.1055	0.0740	0.0866	0.1064	0.0799	0.0787
t (15)	50	0.0827	0.0720	0.0675	0.0778	0.1310	**0.1675**	**0.1443**	0.1116	0.1271	0.0821	0.1120	0.1288	0.0996	0.0975
	80	0.0888	0.0731	0.0874	0.0817	0.1750	**0.2009**	**0.1788**	0.1363	0.1577	0.0866	0.1370	0.1668	0.1175	0.1157
	100	0.1060	0.0743	0.0858	0.0840	0.1902	**0.2105**	**0.1912**	0.1414	0.1667	0.0924	0.1498	0.1758	0.1268	0.1316
	20	0.8865	0.8396	0.8806	0.8662	0.8607	**0.9126**	0.8590	0.7660	0.8182	0.6474	0.8693	**0.8886**	0.8422	0.8300
	30	0.9653	0.9419	0.9630	0.9615	0.9544	**0.9758**	0.9597	0.8460	0.9127	0.6894	0.9594	**0.9664**	0.9546	0.9512
Cauchy (0, 1)	50	0.9966	0.9947	0.9971	0.9954	0.9960	**0.9984**	0.9958	0.9075	0.9665	0.7452	0.9967	0.9966	**0.9972**	0.9968
	80	**1.0000**	0.9997	**1.0000**	0.9999	0.9999	**1.0000**	0.9999	0.9451	0.9796	0.8854	0.9999	0.9999	**1.0000**	**1.0000**
	100	**1.0000**	0.9998	**1.0000**	**1.0000**	0.9999	**1.0000**	**1.0000**	0.9536	0.9848	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
	20	0.1071	0.0873	0.0963	0.0889	**0.1499**	**0.1790**	0.1397	0.1320	0.1423	0.0921	0.1186	0.1421	0.1073	0.0987
	30	0.1208	0.0907	0.1184	0.1151	0.1884	**0.2259**	**0.1932**	0.1653	0.1774	0.1043	0.1462	0.1875	0.1305	0.1293
Logistic (0, 1)	50	0.1620	0.1072	0.1403	0.1270	0.2606	**0.2957**	**0.2672**	0.1870	0.2225	0.1215	0.1892	0.2436	0.1970	0.1866
	80	0.2152	0.1414	0.1838	0.1778	0.3390	**0.3919**	**0.3690**	0.2161	0.3078	0.1266	0.2776	0.3296	0.2623	0.2629
	100	0.2293	0.1548	0.2127	0.1912	0.3834	**0.4361**	**0.4026**	0.2238	0.3360	0.1275	0.2972	0.3753	0.3037	0.3118

Table 8: Monte Carlo power comparisons from long-tailed symmetric alternative distributions at $\alpha = 0.05$.

Distribution	n	Symmetric long-tailed alternative distributions: set 2 (2 of 2)													
		Selected EDF-based tests				Selected moment-based tests						SW tests		Proposed tests	
		AD	LL	CVM	H_n	JB	RJB	b_2	$\sqrt{b_1}$	DP	SEELR	SW	SF	CSELR	SRELR
	20	0.2705	0.2190	0.2697	0.2361	0.3080	**0.3852**	0.2914	0.2511	0.2974	0.1566	0.2593	**0.3092**	0.2373	0.2272
	30	0.3701	0.2932	0.3705	0.3363	0.3968	**0.5071**	0.4127	0.3017	0.3740	0.1741	0.3610	**0.4280**	0.3439	0.3360
Double-exp (0, 1)	50	0.5510	0.4327	0.5327	0.5291	0.5481	**0.6798**	0.5905	0.3554	0.5071	0.1759	0.5147	**0.5956**	0.5320	0.5238
	80	0.7392	0.6121	0.7385	0.7197	0.7212	**0.8350**	0.7549	0.3964	0.6509	0.1854	0.7112	**0.7665**	0.7265	0.7343
	100	0.8207	0.7026	0.8256	0.8103	0.8056	**0.8956**	0.8362	0.4097	0.7353	0.1889	0.8024	**0.8517**	0.8194	0.8208
	20	0.2715	0.2153	0.2709	0.2468	0.2983	**0.3826**	0.2874	0.2576	0.2979	0.1581	0.2577	**0.3166**	0.2296	0.2221
	30	0.3685	0.2934	0.3627	0.3426	0.3993	**0.5058**	0.4050	0.3056	0.3743	0.1779	0.3531	**0.4230**	0.3428	0.3388
Double-exp (0, 4)	50	0.5529	0.4303	0.5409	0.5316	0.5540	**0.6775**	0.5807	0.3476	0.5089	0.1677	0.5213	**0.6004**	0.5223	0.5260
	80	0.7390	0.6142	0.7280	0.7196	0.7220	**0.8278**	0.7606	0.3930	0.6624	0.1817	0.7050	**0.7784**	0.7415	0.7339
	100	0.8216	0.7000	0.8165	0.8088	0.8020	**0.8946**	0.8330	0.4078	0.7321	0.1777	0.7950	**0.8448**	0.8124	0.8160
	20	0.1961	0.1481	0.1822	0.1620	0.2401	**0.2964**	0.2352	0.2235	0.2460	0.1437	0.2041	**0.2510**	0.1865	0.1837
	30	0.2481	0.1760	0.2281	0.2222	0.3410	**0.3959**	**0.3433**	0.2777	0.3223	0.1748	0.2781	0.3248	0.2600	0.2576
Tukey (0, 1, −0.10, −0.10)	50	0.3552	0.2513	0.3273	0.3143	0.4744	**0.5397**	**0.4891**	0.3337	0.4347	0.1936	0.3894	0.4614	0.3897	0.3873
	80	0.4954	0.3475	0.4564	0.4403	0.6216	**0.6864**	**0.6492**	0.3889	0.5735	0.2148	0.5445	0.6105	0.5580	0.5481
	100	0.5774	0.4237	0.5196	0.5190	0.7041	**0.7545**	**0.7216**	0.4105	0.6452	0.2160	0.6218	0.7029	0.6476	0.6464
	20	0.2498	0.1870	0.2383	0.2242	**0.3104**	**0.3640**	0.2911	0.2804	0.3066	0.1778	0.2596	0.3033	0.2279	0.2304
	30	0.3363	0.2388	0.3084	0.2936	0.4120	**0.4755**	0.4167	0.3278	0.3977	0.2163	0.3562	**0.4186**	0.3408	0.3341
Tukey (0, 1, −0.15, −0.15)	50	0.4643	0.3452	0.4325	0.4149	0.5798	**0.6354**	**0.6004**	0.4063	0.5356	0.2375	0.5022	0.5766	0.4995	0.4921
	80	0.6391	0.4881	0.6017	0.5807	0.7335	**0.8018**	**0.7645**	0.4779	0.6899	0.2555	0.6828	0.7342	0.6873	0.6867
	100	0.7253	0.5667	0.6843	0.6747	0.8088	**0.8629**	**0.8342**	0.5127	0.7638	0.2645	0.7557	0.8149	0.7735	0.7837
	20	0.3752	0.2936	0.3461	0.3285	**0.4250**	**0.4838**	0.3993	0.3649	0.4198	0.2567	0.3731	0.4273	0.3396	0.3333
	30	0.4841	0.3878	0.4661	0.4234	0.5574	0.6235	**0.5710**	0.4542	0.5344	0.3127	0.4988	**0.5686**	0.4892	0.4772
Tukey (0, 1, −0.25, −0.25)	50	0.6723	0.5478	0.6456	0.6375	0.7379	**0.7970**	**0.7512**	0.5281	0.7003	0.3260	0.6903	0.7482	0.6903	0.6907
	80	0.8450	0.7406	0.8169	0.8089	0.8797	**0.9197**	**0.8988**	0.6075	0.8503	0.3610	0.8614	0.8934	0.8696	0.8649
	100	0.9069	0.8113	0.8863	0.8953	0.9326	**0.9585**	**0.9447**	0.6434	0.9044	0.9252	0.9128	0.9390	0.9252	0.9279

Table 9: Monte Carlo power comparisons from asymmetric alternative distributions at $\alpha = 0.05$.

Distribution	n	Asymmetric alternative distributions: set 3 (1 of 1)													
		Selected EDF-based tests				Selected moment-based tests						SW tests		Proposed tests	
		AD	LL	CVM	H_n	JB	RJB	b_2	$\sqrt{b_1}$	DP	SEELR	SW	SF	CSELR	SRELR
	20	0.4656	0.3111	0.4130	**0.5309**	0.4105	0.3909	0.2606	0.4653	0.3664	**0.5680**	0.5250	0.5045	0.2754	0.2401
	30	0.6542	0.4684	0.5967	0.7111	0.5720	0.5478	0.3508	0.6650	0.5136	**0.8123**	**0.7516**	0.7103	0.3793	0.3325
Gamma (2, 1)	50	0.8905	0.6973	0.8461	0.9068	0.8240	0.7711	0.4895	0.8915	0.6949	**0.9693**	**0.9469**	0.9266	0.5671	0.4837
	80	0.9878	0.8958	0.9689	0.9857	0.9739	0.9434	0.6363	0.9859	0.8386	**0.9991**	**0.9980**	0.9962	0.7557	0.6737
	100	0.9977	0.9528	0.9897	0.9972	0.9948	0.9845	0.6997	0.9976	0.8796	**0.9999**	**0.9999**	0.9995	0.8307	0.7590
	20	0.1359	0.0941	0.1147	0.1805	0.1234	0.1201	0.0909	0.1405	0.1225	**0.1725**	**0.1548**	0.1452	0.1118	0.1029
	30	0.1929	0.1283	0.1633	0.2343	0.1706	0.1519	0.1144	0.2151	0.1614	**0.2818**	**0.2425**	0.2015	0.1262	0.1183
Weibull (2 ,1)	50	0.3110	0.2062	0.2493	0.3379	0.2656	0.2450	0.1229	0.3670	0.2529	**0.5136**	**0.4111**	0.3516	0.1613	0.1237
	80	0.4869	0.3125	0.4116	0.5063	0.4400	0.3808	0.1413	0.5803	0.3914	**0.7662**	**0.6604**	0.5883	0.2080	0.1597
	100	0.6080	0.3930	0.5119	0.6295	0.5661	0.4818	0.1529	0.6976	0.4703	**0.8703**	**0.7940**	0.7185	0.2216	0.1753
	20	0.2443	0.1783	0.2156	**0.3008**	0.1992	0.2003	0.1300	0.2497	0.2212	**0.2915**	0.2764	0.2600	0.1576	0.1297
	30	0.3563	0.2464	0.3238	**0.4197**	0.2956	0.2828	0.1584	0.3710	0.3343	**0.4659**	0.4157	0.3955	0.1938	0.1612
SN (0, 1, 5)	50	0.5852	0.4036	0.5246	0.6397	0.4815	0.4390	0.2084	0.6077	0.4672	**0.7048**	**0.6677**	0.6331	0.2642	0.2099
	80	0.8152	0.6093	0.7556	0.8268	0.7366	0.6619	0.2529	0.8291	0.7346	**0.9028**	**0.8874**	0.8533	0.3769	0.2935
	100	0.8987	0.7235	0.8487	0.9059	0.8484	0.7889	0.2847	0.9204	0.8004	**0.9523**	**0.9517**	0.9303	0.4283	0.3323
	20	0.9693	0.9353	0.9632	**0.9769**	0.9477	0.9446	0.8649	0.9526	0.9346	0.9514	**0.9705**	0.9693	0.8690	0.8444
	30	**0.9974**	0.9882	0.9958	0.9972	0.9905	0.9910	0.9567	0.9884	0.9708	0.9782	0.9961	**0.9974**	0.9680	0.9585
SC (0, 2, 5)	50	**1.0000**	0.9997	**1.0000**	**1.0000**	**1.0000**	0.9999	0.9958	0.9972	0.9886	0.9857	0.9999	**1.0000**	0.9981	0.9972
	80	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	0.9998	0.9985	0.9900	0.9934	**1.0000**	**1.0000**	**1.0000**	**1.0000**
	100	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	0.9999	0.9990	0.9912	0.9982	**1.0000**	**1.0000**	**1.0000**	**1.0000**
	20	0.2558	0.1745	0.2298	0.1162	0.0767	0.0744	0.0887	0.1157	0.1107	**0.3243**	**0.3062**	0.2147	0.1905	0.1822
	30	0.4282	0.2654	0.3672	0.2107	0.0918	0.0815	0.0896	0.1749	0.1678	**0.5164**	**0.5185**	0.3676	0.3569	0.3116
Beta (2, 1)	50	0.7142	0.4572	0.6174	0.4376	0.1992	0.1175	0.1094	0.3200	0.3021	**0.7994**	**0.8411**	0.7026	0.7051	0.5928
	80	0.9349	0.7011	0.8569	0.7309	0.6129	0.2429	0.1765	0.5645	0.5511	**0.9641**	**0.9871**	0.9538	0.9478	0.8971
	100	0.9817	0.8204	0.9362	0.8540	0.8353	0.4430	0.2243	0.6914	0.6699	0.9909	**0.9983**	**0.9935**	0.9840	0.9627
	20	0.1447	0.1057	0.1348	0.0702	0.0615	0.0623	0.0687	0.0915	0.0871	**0.1855**	**0.1715**	0.1272	0.1064	0.1091
	30	0.2313	0.1587	0.2051	0.1181	0.0793	0.0727	0.0659	0.1335	0.1237	**0.3157**	**0.2700**	0.1985	0.1844	0.1735
Beta (3, 1.5)	50	0.4311	0.2718	0.3670	0.2188	0.1277	0.0935	0.0716	0.2395	0.2022	**0.5751**	**0.5390**	0.3935	0.4089	0.3435
	80	0.6974	0.4347	0.5793	0.4296	0.3340	0.1670	0.0940	0.4359	0.3654	**0.8381**	**0.8412**	0.7221	0.7253	0.6313
	100	0.8207	0.5347	0.7112	0.5502	0.5358	0.2722	0.1220	0.5671	0.4855	**0.9270**	**0.9370**	0.8628	0.8347	0.7700

Table 10: Ranking of tests using average powers computed from the values in Tables 5 and 6 for set 1 of alternative distributions.

Ranking	Power rankings under symmetric short-tailed alternative distributions		
	Small sample sizes n = 20–50	Moderate sample sizes n = 80–100	Overall sample sizes n = 20–100
1	SRELR, CSELR	SRELR, CSELR	SRELR, CSELR
2	DP	DP	DP
3	SW, b_2	b_2	SW, b_2
4	SEELR, AD	SW	AD
5	H_n, CVM	AD	SEELR
6	SF	SF	H_n, CVM, SF
7	LL	SEELR	LL
8	$\sqrt{b_1}$, JB, RJB	H_n	JB
9		CVM	RJB
10		LL	$\sqrt{b_1}$
11		JB	
12		RJB, $\sqrt{b_1}$	

Table 11: Ranking of tests using average powers computed from the values in Tables 7 and 8 for set 2 of alternative distributions.

Ranking	Power rankings under symmetric long-tailed alternative distributions		
	Small sample sizes $n = 20 - 50$	Moderate sample sizes $n = 80 - 100$	Overall sample sizes $n = 20 - 100$
1	RJB	RJB	RJB
2	SF, b_2, JB	b_2, SF, JB	b_2, SF, JB
3	DP	SRELR, CSELR	DP
4	SW	DP, SW	SW, CSELR, SRELR
5	CSELR, SRELR, AD	AD	AD
6	CVM	CVM, H_n	CVM, H_n
7	$\sqrt{b_1}$, H_n	LL	$\sqrt{b_1}$, LL
8	LL	$\sqrt{b_1}$	SEELR
9	SEELR	SEELR	

Table 12: Ranking of tests using average powers computed from the values in Table 9 for set 3 of alternative distributions.

Ranking	Power rankings under asymmetric alternative distributions		
	Small sample sizes $n = 20 - 50$	Moderate sample sizes $n = 80 - 100$	Overall sample sizes $n = 20 - 100$
1	SEELR	SEELR	SEELR
2	SW	SW	SW
3	SF, AD	SF	SF
4	H_n, CVM	AD	AD
5	$\sqrt{b_1}$	CVM, H_n	CVM, H_n
6	LL, CSELR, DP	$\sqrt{b_1}$	$\sqrt{b_1}$
7	JB	JB	JB
8	RJB	LL, CSELR	LL, CSELR
9	SRELR	DP	DP
10	b_2	SRELR	SRELR
11		RJB	RJB
12		b_2	b_2

year 1910 to 1972. The dataset has been extensively used in various statistical applications; see, for example, Thaler [42], Carmichael [43], Tukey [44], and Parzen [45] to illustrate and compare various statistical techniques. The snowfall dataset is presented as follows: 126.4, 82.4, 78.1, 51.1, 90.9, 76.2, 104.5, 87.4, 110.5, 25.0, 69.3, 53.5, 39.8, 63.6, 46.7, 72.9, 79.6, 83.6, 80.7, 60.3, 79.0, 74.4, 49.6, 54.7, 71.8, 49.1, 103.9, 51.6, 82.4, 83.6, 77.8, 79.3, 89.6, 85.5, 58.0, 120.7, 110.5, 65.4, 39.9, 40.1, 88.7, 71.4, 83.0, 55.9, 89.9, 84.8, 105.2, 113.7, 124.7, 114.5, 115.6, 102.4, 101.4, 89.8, 71.5, 70.9, 98.3, 55.5, 66.1, 78.4, 102.5, 97.0, 110.0.

The snowfall data is well known to be consistent with the normal distribution. We plotted a histogram and a Q-Q plot in order to examine the hypothesis for the normality of the snowfall data (see Figure 3).

From the plots, it is clearly visible that the snowfall data are consistent with a normal distribution. Following the ideas introduced by Stigler [46], we conducted a bootstrap

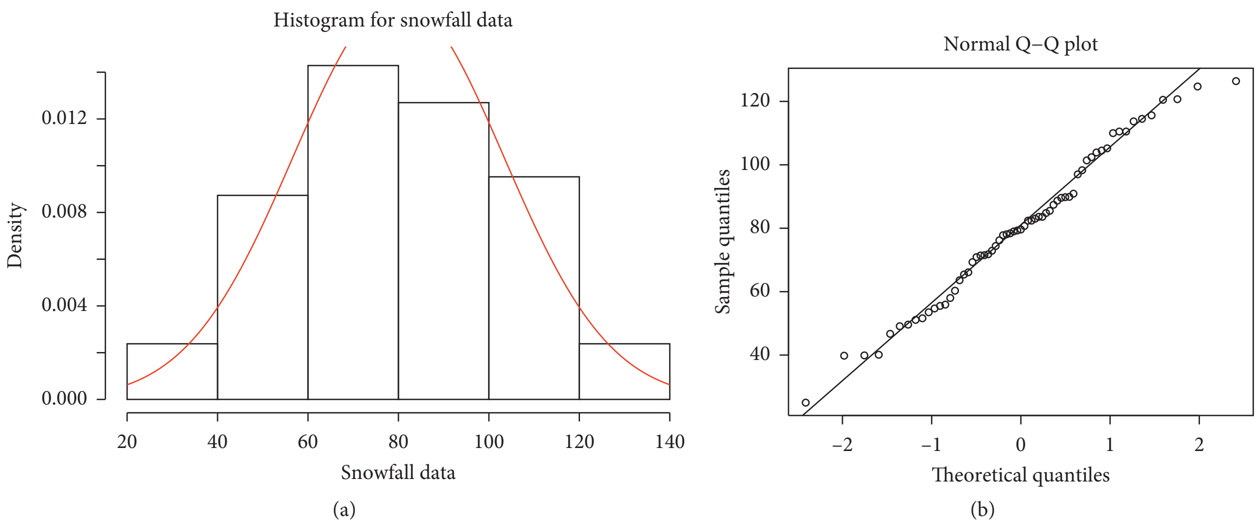

FIGURE 3: Histogram and Q-Q plots for snowfall data.

type study to empirically examine the proposed test based on the two statistics, CSELR and SRELR. The approach was to use a sample of size 60 by randomly selecting from the snowfall data and then test for normality at 0.05 level of significance. We repeated this strategy 10000 times, and the bootstrap type procedure showed that the proposed CSELR test had a p value of 0.7755, while the SRELR had a p value of 0.1451. In order to further examine the normality of the snowfall data, we repeated the bootstrap type study using the AD, CVM, JB, and SW tests. The p values that were obtained, that is, 0.6862 for the AD test, 0.6921 for the CVM test, 0.5702 for the JB test, and 0.6650 for the SW test were all suggestive for one to conclude that the snowfall data are indeed normally distributed. Thus, the p values obtained from the traditional tests as well as our proposed tests show to be reliable in illustrating the normality of the snowfall data. Thus, our proposed test statistics have demonstrated that they are indeed applicable when applied on some real-life data.

5. Conclusion

By utilizing the EL methodology and exploiting the mathematical properties and different forms of transforming the normal distribution, we have developed simple and powerful tests for normality against symmetric alternatives. The proposed tests are consistent and control type I error very well, which is consistent with what has been reported in other studies which looked at EL-based GoF tests (see, for example, [8, 12, 18]). They outperformed other common traditional tests under symmetric short-tailed alternatives. The proposed tests also performed quite well under symmetric long-tailed alternatives where they were found to be comparable to the SW test and outperformed all the considered EDF tests. The application of our proposed tests on real data revealed the applicability as well as the robustness of the proposed tests in practice. It would be desirable to develop an ELR-based test for normality that outperforms the classical tests under most alternative distributions that occur in practice. This might be the case after certain modifications and improvements that include further exploring the EL methodology as well as other forms of characterizing the normal distribution. The researchers are currently looking at exploiting the use of EDF in developing an empirical likelihood moment-based EDF test for normality. Thus, combining the characterization of EDF-based tests and EL omnibus tests can potentially improve power under small to moderate sample sizes.

Appendix

```
#Required packages
library(emplik)
library(zipfR)
#Generate standardized data
genedata < −function(n){
#Generate data
s < −runif(n, 0, 1)
x1 < −s
for(k in 1 : n){
s[k] < −(x1[k] − mean(x1)) * sqrt(n/(n −1))/(sd(x1
[−k]))
}
return(s)
}
#Generate transformed data
genedata1 < −function(n) {
v< −genedata(n)
x < −pnorm(v, 0, 1)
return(x)}
```

```
#Moment function for uniform distribution
momentFU < -function(k, a, b){
z < -(b^(k + 1) - a^(k + 1))/((k + 1) * (b - a))
}
#Compute test statistic
teststatistic < -function(x)
{
#CUSUM-type statistic
k3 = el.test(x**3, m3)$-2LLR
k5 = el.test(x**5, m5)$-2LLR
return(max(k3, k5))
#Shiryaev–Roberts statistic
#k3 = exp(el.test(x**3, m3)$-2LLR
#k5 = exp(el.test(x**5, m5)$-2LLR
#return(sum(k3, k5))
}
n < -50 #sample size
a < -0 #lower limit for moment function
b < -1 #upper limit for moment function
#Critical values
#Critical values for CUSUM statistic
CriticalValue < -0.2685595 #n = 50
###########################################
#Critical values for SR statistic
#CriticalValue < -2.459526 #n = 50
MC < -10000 #number of replications
power < - c()
m3 = momentFU(3, a, b)
m5 = momentFU(5, a, b)
for(i in 1 : MC) {
x < -genedata1(n)
power[i] < -teststatistic(x) }
# Power for the test under alternative
PowerELR = (mean(power > CriticalValue))
```

Data Availability

The data used to demonstrate the applicability of our proposed tests in practice are presented in this article and can also be obtained from respective authors cited in the "Real Data Study" section. All other data were simulated using R and the source code is available in the Appendix.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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