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Research Article

Reliability Estimation for the Remained Stress-Strength Model under the Generalized Exponential Lifetime Distribution

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A stress-strength reliability model compares the strength and stresses on a certain system; it is used not only primarily in reliability engineering and quality control but also in economics, psychology, and medicine. In this paper, a novel extension of stress-strength models is presented. The mew model is applied under the generalized exponential distribution. The maximum likelihood estimator, asymptotic distribution, and Bayesian estimation are obtained. A comprehensive simulation study along with real data analysis is performed for illustrating the importance of the new stress-strength model.

1. Introduction

Stress-strength reliability analysis is a statistical analysis of the interference of of the strength of the component and the stresses placed on the component. The stress-strength reliability analysis is a statistical tool used in reliability engineering.

In a stress-strength reliability model, both strength and stresses are considered as separate random variables. Stress experienced by a certain component is usually presented by the random variable X and the strength of the same component is presented by the ramdom variable Y. A situation in which $X > Y$ is one in which the stress is greater than the strength, and then, the component fails.

The abovementionedprobability model can be expressed as $R = \Pr(X > Y)$ and then called as the stress-strength quantity. The stress-strength reliability model has various applications in many fields such as reliability, quality control, and engineering. For more details in this matter, see the work of Kotz et al. [1] and Ventura and Racugno [2]. Rezaei et al. [3] presented a list of probability distributions used under the stress-strength reliability model. Recently, Rasekhi et al. [4] presented a Bayesian and the classical inference of reliability in multicomponent stress-strength under the generalized logistic model. Saber and Yousof [5] investigated the Bayesian and the classical inference for the generalized stress-strength parameter under generalized logistic distribution.

We suppose that we know these two components have been worked till a known time, and then, we are going to have some inferences on R. For this case, Saber and Khorshidian [6] introduce the conditional stress-strength model $R^{|a,b}$:

$$R^{|a,b} = P(X > Y | X > a, Y > b). \tag{1}$$

When independent random variables X and Y are continues, $R^{|a,b}$ can be computed by the following equation:

$$R^{|a,b} = \begin{cases} \dfrac{1 - F_Y(a) - \int_a^{+\infty} F_X(y) f_Y(y) \mathrm{d}y}{[1 - F_X(a)][1 - F_Y(b)]}\Big|_{a=b}, \\ \dfrac{1 - F_Y(b) - \int_b^{+\infty} F_X(y) f_Y(y) \mathrm{d}y}{[1 - F_X(a)][1 - F_Y(b)]}\Big|_{a<b}, \\ \dfrac{\int_a^{+\infty} F_Y(x) f_X(x) \mathrm{d}x - F_Y(b)(1 - F_X(a))}{[1 - F_X(a)][1 - F_Y(b)]}\Big|_{a>b}. \end{cases} \tag{2}$$

Saber and Khorshidian [6] studied this model for the case of exponential distributed components under the nonparametric case, and the Bayesian estimation is ignored. However, in this work, we study the conditional stress-strength model for two-parameter generalized exponential (GE) distributed components, and the Bayesian estimation is considered. The GE distribution can be used as an alternative to gamma, Weibull, and log-normal distributions. For more related studies, see the work of Gupta and Kundu [7] and Kundu (2002) and Rao [8].

The probability density function and cumulative distribution function of GE distribution are, respectively,

$$\begin{aligned} f_X(x) &= \alpha\lambda e^{-\lambda x}\left(1 - e^{-\lambda x}\right)^{\alpha-1}, \\ F_X(x) &= \left(1 - e^{-\lambda x}\right)^{\alpha}, \end{aligned} \tag{3}$$

where $x \geq 0$ and $\alpha > 0$ are the shape parameters and $\lambda > 0$ is the scale parameter.

The rest of the paper is organized as follows. We devote Section 2 to study $R^{|a,b}$ in case of GE distribution. In Section 3, the ML estimator of quantity $R^{|a,b}$ and its corresponding asymptotic distribution and confidence interval are provided. A simulation study is presented in Section 4, and Section 5 has been devoted to applying a real dataset to the recommended model.

2. Conditional Stress-Strength Model for GE Distribution

In this section, quantity (2) is computed when distribution of components is GE.

Theorem 1. *We suppose random variables X and Y are independent and $X \sim GE(\alpha_1, \lambda)$ and $Y \sim GE(\alpha_2, \lambda)$; then,*

$$R^{|a,b} = \begin{cases} \dfrac{1 - \left(1 - e^{-\lambda b}\right)^{\alpha_2} - \left(\alpha_2/(\alpha_1 + \alpha_2)\right)}{\left[1 - \left(1 - e^{-\lambda a}\right)^{\alpha_1}\right]\left[1 - \left(1 - e^{-\lambda b}\right)^{\alpha_2}\right]}\Big|_{a=b}, \\ \dfrac{1 - \left(1 - e^{-\lambda b}\right)^{\alpha_2} - \left(\alpha_2/(\alpha_1 + \alpha_2)\right)\left[1 - \left(1 - e^{-\lambda b}\right)^{\alpha_2+\alpha_1}\right]}{\left[1 - \left(1 - e^{-\lambda a}\right)^{\alpha_1}\right]\left[1 - \left(1 - e^{-\lambda b}\right)^{\alpha_2}\right]}\Big|_{a<b}, \\ \dfrac{\left(\alpha_2/(\alpha_1 + \alpha_2)\right)\left[1 - \left(1 - e^{-\lambda a}\right)^{\alpha_1+\alpha_2}\right] - \left\{\left(1 - e^{-\lambda b}\right)^{\alpha_2}\left[1 - \left(1 - e^{-\lambda a}\right)^{\alpha_1}\right]\right\}}{\left[1 - \left(1 - e^{-\lambda a}\right)^{\alpha_1}\right]\left[1 - \left(1 - e^{-\lambda b}\right)^{\alpha_2}\right]}\Big|_{a>b}. \end{cases} \tag{4}$$

Proof. Let

$$\begin{aligned} f_X(y) &= \alpha_1\lambda e^{-\lambda y}\left(1 - e^{-\lambda y}\right)^{\alpha_1-1}, \\ f_Y(y) &= \alpha_2\lambda e^{-\lambda y}\left(1 - e^{-\lambda y}\right)^{\alpha_2-1}, \\ F_X(y) &= \left(1 - e^{-\lambda y}\right)^{\alpha_1}, \\ F_Y(y) &= \left(1 - e^{-\lambda y}\right)^{\alpha_2}, \\ F_X(a) &= \left(1 - e^{-\lambda a}\right)^{\alpha_1}, \\ F_Y(b) &= \left(1 - e^{-\lambda b}\right)^{\alpha_2}. \end{aligned} \tag{5}$$

Substitute the last six equations in equation (2). Let $a = b$; then,

$$R^{|a,b} = \frac{1 - F_Y(a) - \int_a^{+\infty} F_X(y) f_Y(y) \mathrm{d}y}{[1 - F_X(a)][1 - F_Y(b)]}. \tag{6}$$

Then,

$$\Rightarrow R^{|a,b} = \frac{1 - F_Y(a) - \int_a^{+\infty} \left(1 - e^{-\lambda y}\right)^{\alpha_1} \alpha_2\lambda e^{-\lambda y}\left(1 - e^{-\lambda y}\right)^{\alpha_2-1} \mathrm{d}y}{\left(1 - \left(1 - e^{-\lambda b}\right)^{\alpha_1}\right)\left(1 - \left(1 - e^{-\lambda b}\right)^{\alpha_2}\right)},$$

$$\Rightarrow R^{|a,b} = \frac{1 - F_Y(a) - \int_a^{+\infty} \alpha_2 \lambda e^{-\lambda y}\left(1 - e^{-\lambda y}\right)^{\alpha_1+\alpha_2-1} \mathrm{d}y}{\left(1 - \left(1 - e^{-\lambda b}\right)^{\alpha_1}\right)\left(1 - \left(1 - e^{-\lambda b}\right)^{\alpha_2}\right)}, \Rightarrow R^{|a,b}$$
$$= \frac{1 - F_Y(a) - \left(\alpha_2/\left(\alpha_1+\alpha_2\right)\right) \int_a^{+\infty} \left(\alpha_1+\alpha_2\right) \lambda e^{-\lambda y}\left(1 - e^{-\lambda y}\right)^{\alpha_1+\alpha_2-1} \mathrm{d}y}{\left(1 - \left(1 - e^{-\lambda b}\right)^{\alpha_1}\right)\left(1 - \left(1 - e^{-\lambda b}\right)^{\alpha_2}\right)}, \Rightarrow R^{|a,b}$$
$$= \frac{1 - F_Y(a) - \left(\alpha_2/\left(\alpha_1+\alpha_2\right)\right)\left(1 - F_W(a)\right)}{\left(1 - \left(1 - e^{-\lambda b}\right)^{\alpha_1}\right)\left(1 - \left(1 - e^{-\lambda b}\right)^{\alpha_2}\right)}. \tag{7}$$

In the abovementioned result, it is noted that $W \sim \mathrm{GE}(\alpha_1 + \alpha_2, \lambda)$; thus,

$$F_W(a) = \left(1 - e^{-\lambda a}\right)^{\alpha_1+\alpha_2}. \tag{8}$$

Since $a = b$, the proof of Theorem 1 is completed.

If $a < b$, then

$$R^{|a,b} = \frac{1 - F_Y(b) - \int_b^{+\infty} F_X(y) f_Y(y) \mathrm{d}y}{\left[1 - F_X(a)\right]\left[1 - F_Y(b)\right]}. \tag{9}$$

The dominator of $R^{|a,b}$ in this case is exactly the same as in the first case with substitution of b instead of a. Therefore,

$$R^{|a,b} = \frac{1 - F_Y(b) - \left(\alpha_2/\left(\alpha_1+\alpha_2\right)\right)\left[1 - F_W(b)\right]}{\left(1 - \left(1 - e^{-\lambda a}\right)^{\alpha_1}\right)\left(1 - \left(1 - e^{-\lambda b}\right)^{\alpha_2}\right)}. \tag{10}$$

Then,

$$R^{|a,b} = \frac{1 - \left(1 - e^{-\lambda b}\right)^{\alpha_2} - \left(\alpha_2/\left(\alpha_1+\alpha_2\right)\right)\left(1 - \left(1 - e^{-\lambda b}\right)^{\alpha_2+\alpha_1}\right)}{\left(1 - \left(1 - e^{-\lambda a}\right)^{\alpha_1}\right)\left(1 - \left(1 - e^{-\lambda b}\right)^{\alpha_2}\right)}. \tag{11}$$

Eventually, if $a > b$, we can calculate $R^{|a,b}$ like previous ones.

$$R^{|a,b} = \frac{\int_a^{+\infty} F_Y(x) f_X(x) \mathrm{d}x - F_Y(b)\left(1 - F_X(a)\right)}{\left[1 - F_X(a)\right]\left[1 - F_Y(b)\right]}. \tag{12}$$

Then,

$$R^{|a,b} = \frac{\int_a^{+\infty} \left(1 - e^{-\lambda y}\right)^{\alpha_2} \alpha_1 \lambda e^{-\lambda y}\left(1 - e^{-\lambda y}\right)^{\alpha_1-1} \mathrm{d}y - F_Y(b)\left(1 - F_X(a)\right)}{\left[1 - F_X(a)\right]\left[1 - F_Y(b)\right]}, \tag{13}$$

which can be expressed as

$$R^{|a,b} = \frac{\left(\alpha_1/\left(\alpha_1+\alpha_2\right)\right)\left[1 - F_W(a)\right] - F_Y(b)\left(1 - F_X(a)\right)}{\left[1 - F_X(a)\right]\left[1 - F_Y(b)\right]}. \tag{14}$$

Below, we derive the maximum likelihood estimation (MLE) of the $R^{|a,b}$ model; and hence, the asymptotic distribution of those is presented in order to constructing the corresponding confidence interval.

Let $X_1, X_2, \ldots, X_m$ be a random sample of size m of $\mathrm{GE}(\alpha_1, \lambda)$ and $Y_1, Y_2, \ldots, Y_n$ be a random sample of size n of $\mathrm{GE}(\alpha_2, \lambda)$ such that X and Y are independent. Then, the likelihood function can be expressed as

$$\mathbf{L} = \alpha_1^m \alpha_2^n \lambda^{m+n} e^{-\lambda\left(\sum_{i=1}^m x_i + \sum_{j=1}^n y_j\right)} \times \left[\prod_{i=1}^m \left(1 - e^{-\lambda x_i}\right)\right]^{\alpha_1-1} \left[\prod_{j=1}^n \left(1 - e^{-\lambda y_j}\right)\right]^{\alpha_2-1}. \tag{15}$$

Then, the log-likelihood function is given by

$$\mathbf{L}_{i,j}^{m,n} = m \ln \alpha_1 + n \ln \alpha_2 + (m+n) \ln \lambda - \lambda\left(\sum_{i=1}^m x_i + \sum_{j=1}^n y_j\right) + \left(\alpha_1 - 1\right)\sum_{i=1}^m \ln\left(1 - e^{-\lambda x_i}\right) + \left(\alpha_2 - 1\right)\sum_{j=1}^n \ln\left(1 - e^{-\lambda y_j}\right). \tag{16}$$

Therefore, the maximum likelihood estimator of parameters can be obtained by solving $(\partial \mathbf{L}_{i,j}^{m,n}/\partial\lambda) = 0$, $(\partial \mathbf{L}_{i,j}^{m,n}/\partial\alpha_1) = 0$ and $(\partial \mathbf{L}_{i,j}^{m,n}/\partial\alpha_2) = 0$. A simple computation shows

$$\begin{aligned}
\frac{\partial \mathbf{L}_{i,j}^{m,n}}{\partial \alpha_1} &= \frac{m}{\alpha_1} + \sum_{i=1}^m \ln\left(1 - e^{-\lambda x_i}\right), \\
\frac{\partial L}{\partial \alpha_2} &= \frac{n}{\alpha_2} + \sum_{i=1}^m \ln\left(1 - e^{-\lambda x_i}\right), \\
\frac{\partial \mathbf{L}_{i,j}^{m,n}}{\partial \lambda} &= \frac{m+n}{\lambda} - \left(\sum_{i=1}^m x_i + \sum_{j=1}^n y_j\right) + \left(\alpha_1 - 1\right)\sum_{i=1}^m \frac{x_i e^{-\lambda x_i}}{1 - e^{-\lambda x_i}} \\
&\quad + \left(\alpha_2 - 1\right)\sum_{j=1}^n \frac{y_j e^{-\lambda y_j}}{1 - e^{-\lambda y_j}}.
\end{aligned} \tag{17}$$

Therefore, we have

$$\alpha_1 = -\frac{m}{\sum_{i=1}^{m} \ln\left(1 - e^{-\lambda x_i}\right)}, \tag{18}$$

$$\alpha_2 = -\frac{n}{\sum_{j=1}^{n} \ln\left(1 - e^{-\lambda y_j}\right)}. \tag{19}$$

Equations (18) and (19) depend on unknown parameter λ. We substitute equations (18) and (19) in $\partial \mathbf{L}_{i,j}^{m,n}/\partial\lambda$, and we can find $\widehat{\lambda}$ by solving the following nonlinear equation:

$$\frac{\partial \mathbf{L}_{i,j}^{m,n}}{\partial\lambda} = \frac{m+n}{\lambda} - \left(\sum_{i=1}^{m} x_i + \sum_{j=1}^{n} y_j\right) - \left[\frac{m}{\sum_{i=1}^{m} \ln\left(1 - e^{-\lambda x_i}\right)} + 1\right] \sum_{i=1}^{m} \frac{x_i e^{-\lambda x_i}}{1 - e^{-\lambda x_i}} - \left[\frac{n}{\sum_{j=1}^{n} \ln\left(1 - e^{-\lambda y_j}\right)} + 1\right] \sum_{j=1}^{n} \frac{y_j e^{-\lambda y_j}}{1 - e^{-\lambda y_j}} = 0. \tag{20}$$

Then, two other parameters are earned by substitution $\widehat{\lambda}$ in equations (18) and (19) as

$$\begin{aligned} \widehat{\alpha}_1 &= -\frac{m}{\sum_{i=1}^{m} \ln\left(1 - e^{-\widehat{\lambda} x_i}\right)}, \\ \widehat{\alpha}_2 &= -\frac{n}{\sum_{j=1}^{n} \ln\left(1 - e^{-\widehat{\lambda} y_j}\right)}. \end{aligned} \tag{21}$$

Therefore, the maximum likelihood estimator of $R^{|a,b}$ becomes

$$\widehat{R}^{|a,b} = \begin{cases} \dfrac{1 - \left(1 - e^{-\widehat{\lambda} b}\right)^{\widehat{\alpha}_2} - \left(\widehat{\alpha}_2 / \left(\widehat{\alpha}_1 + \widehat{\alpha}_2\right)\right)\left[1 - \left(1 - e^{-\widehat{\lambda} b}\right)^{\widehat{\alpha}_2 + \widehat{\alpha}_1}\right]}{\left[1 - \left(1 - e^{-\widehat{\lambda} a}\right)^{\widehat{\alpha}_1}\right]\left[1 - \left(1 - e^{-\widehat{\lambda} b}\right)^{\widehat{\alpha}_2}\right]}\Bigg|_{a=b}, \\[2ex] \dfrac{1 - \left(1 - e^{-\widehat{\lambda} b}\right)^{\widehat{\alpha}_2} - \left(\widehat{\alpha}_2 / \left(\widehat{\alpha}_1 + \widehat{\alpha}_2\right)\right)\left[1 - \left(1 - e^{-\widehat{\lambda} b}\right)^{\widehat{\alpha}_2 + \widehat{\alpha}_1}\right]}{\left[1 - \left(1 - e^{-\widehat{\lambda} a}\right)^{\widehat{\alpha}_1}\right]\left[1 - \left(1 - e^{-\widehat{\lambda} b}\right)^{\widehat{\alpha}_2}\right]}\Bigg|_{a<b}, \\[2ex] \dfrac{\left(\widehat{\alpha}_1 / \left(\widehat{\alpha}_1 + \widehat{\alpha}_2\right)\right)\left[1 - \left(1 - e^{-\widehat{\lambda} a}\right)^{\widehat{\alpha}_1 + \widehat{\alpha}_2}\right] - \left\{\left(1 - e^{-\widehat{\lambda} b}\right)^{\widehat{\alpha}_2}\left[1 - \left(1 - e^{-\widehat{\lambda} a}\right)^{\widehat{\alpha}_1}\right]\right\}}{\left[1 - \left(1 - e^{-\widehat{\lambda} a}\right)^{\widehat{\alpha}_1}\right]\left[1 - \left(1 - e^{-\widehat{\lambda} b}\right)^{\widehat{\alpha}_2}\right]}\Bigg|_{a>b}. \end{cases} \tag{22}$$

In continuation with this section, the asymptotic distribution of $\widehat{\theta} = (\widehat{\lambda}, \widehat{\alpha}_1, \widehat{\alpha}_2)$ and the asymptotic distribution of $R^{|a,b}$ are obtained. The Fisher information matrix of $\theta = (\lambda, \alpha_1, \alpha_2)$ denoted by $J(\theta) = E[I(\theta)]$, where $I(\theta) = [I_{i,j}]_{i,j=1,2,3}$ is the observed information matrix; i.e.,

$$I(\boldsymbol{\theta}) = -\begin{pmatrix} \dfrac{\partial^2 \mathbf{L}_{i,j}^{m,n}(\lambda, \alpha_1, \alpha_2)}{\partial\lambda^2} & \dfrac{\partial^2 \mathbf{L}_{i,j}^{m,n}(\lambda, \alpha_1, \alpha_2)}{\partial\lambda\,\partial\alpha_1} & \dfrac{\partial^2 \mathbf{L}_{i,j}^{m,n}(\lambda, \alpha_1, \alpha_2)}{\partial\lambda\,\partial\alpha_2} \\[2ex] \dfrac{\partial^2 \mathbf{L}_{i,j}^{m,n}(\lambda, \alpha_1, \alpha_2)}{\partial\alpha_1\partial\lambda} & \dfrac{\partial^2 \mathbf{L}_{i,j}^{m,n}(\lambda, \alpha_1, \alpha_2)}{\partial\alpha_1^2} & \dfrac{\partial^2 \mathbf{L}_{i,j}^{m,n}(\lambda, \alpha_1, \alpha_2)}{\partial\alpha_1\partial\alpha_2} \\[2ex] \dfrac{\partial^2 \mathbf{L}_{i,j}^{m,n}(\lambda, \alpha_1, \alpha_2)}{\partial\alpha_2\partial\lambda} & \dfrac{\partial^2 \mathbf{L}_{i,j}^{m,n}(\lambda, \alpha_1, \alpha_2)}{\partial\alpha_2\partial\alpha_1} & \dfrac{\partial^2 \mathbf{L}_{i,j}^{m,n}(\lambda, \alpha_1, \alpha_2)}{\partial\alpha_2^2} \end{pmatrix}. \tag{23}$$

Now, the elements of $J(\theta)$ follow by

$$J_{11} = -E\left[\frac{\partial^2 \mathbf{L}_{i,j}^{m,n}}{\partial\lambda^2}\right], \tag{24}$$

and then,

$$J_{11} = -E\left(-\frac{m+n}{\lambda^2} + (\alpha_1 - 1)\sum_{i=1}^{m}\frac{-x_i^2 e^{-\lambda x_i}}{\left(1-e^{-\lambda x_i}\right)^2} + (\alpha_2 - 1)\sum_{j=1}^{n}\frac{-y_j^2 e^{-\lambda y_j}}{\left(1-e^{-\lambda y_j}\right)^2}\right). \tag{25}$$

It can finally be derived as

$$\begin{aligned}
J_{11} &= \frac{m+n}{\lambda^2} + m(\alpha_1 - 1)\frac{\alpha_1}{\lambda^2}\frac{\partial^2 \mathrm{Beta}(t+1, \alpha_1 - 2)}{\partial^2 t}\Big|_{t=2} \\
&\quad + n(\alpha_2 - 1)\frac{\alpha_2}{\lambda^2}\frac{\partial^2 Beta(t+1, \alpha_2 - 2)}{\partial^2 t}\Big|_{t=2}, \\
J_{12} &= J_{21} = -E\left(\frac{\partial^2 L}{\partial\lambda\,\partial\alpha_1}\right) = -E\left(\sum_{i=1}^{m}\frac{x_i e^{-\lambda x_i}}{1-e^{-\lambda x_i}}\right), \\
&= \frac{m\alpha_1}{\lambda}\frac{\partial \mathrm{Beta}(t+1, \alpha_1 - 1)}{\partial t}\Big|_{t=1}, \\
J_{13} &= J_{31} = -E\left(\frac{\partial^2 L}{\partial\lambda\,\partial\alpha_2}\right) = -E\left(\sum_{j=1}^{n}\frac{y_j e^{-\lambda y_j}}{1-e^{-\lambda y_j}}\right) = \frac{n\alpha_2}{\lambda}\frac{\partial \mathrm{Beta}(t+1, \alpha_2 - 1)}{\partial t}\Big|_{t=1}, \\
J_{22} &= -E\left(\frac{\partial^2 \mathbf{L}_{i,j}^{m,n}}{\partial\alpha_1^2}\right) = -E\left(-\frac{m}{\alpha_1^2}\right) = \frac{m}{\alpha_1^2}, \\
J_{23} &= J_{32} = -E\left(\frac{\partial^2 \mathbf{L}_{i,j}^{m,n}}{\partial\alpha_1\partial\alpha_2}\right) = 0, \\
J_{33} &= -E\left(\frac{\partial^2 \mathbf{L}_{i,j}^{m,n}}{\partial\alpha_2^2}\right) = -E\left(-\frac{n}{\alpha_2^2}\right) = \frac{n}{\alpha_2^2}.
\end{aligned} \tag{26}$$

By characteristics of MLEs, we have $\hat{\theta} \approx N_3(\theta, \Sigma)$ for a large number of n and m, where Σ is the inverse of the Fisher information matrix $I(\theta)$.

$$\begin{aligned}
\mathbf{\Sigma} &= \frac{1}{\det I(\boldsymbol{\theta})}\begin{pmatrix} J_{22}J_{33} - J_{23}J_{32} & J_{13}J_{32} - J_{12}J_{33} & J_{12}J_{23} - J_{13}J_{22} \\ J_{23}J_{31} - J_{21}J_{33} & J_{11}J_{33} - J_{13}J_{31} & J_{13}J_{21} - J_{11}J_{23} \\ J_{21}J_{32} - J_{22}J_{31} & J_{12}J_{31} - J_{11}J_{32} & J_{11}J_{22} - J_{12}J_{21} \end{pmatrix}, \\
\mathbf{\Sigma} &= \frac{1}{\det I(\boldsymbol{\theta})}\begin{pmatrix} J_{22}J_{33} & -J_{12}J_{33} & -J_{13}J_{22} \\ -J_{21}J_{33} & J_{11}J_{33} - J_{13}J_{31} & J_{13}J_{21} \\ -J_{22}J_{31} & J_{12}J_{31} & J_{11}J_{22} - J_{12}J_{21} \end{pmatrix}.
\end{aligned} \tag{27}$$

In Lemma 1, we recall the multivariate Delta method in order to find an asymptotic distribution of $R^{|a,b}$. □

Lemma 1. *Let $\{\mathbf{X}_n\}_{n=1}^{\infty}$ be a sequence of random vector, where $\mathbf{X}_n \longrightarrow \mathbf{N}(\mu, \Sigma)$ in distribution. If the function $\mathbf{g}$ is continuous in the first partial derivatives and $\tau = (\nabla\mathbf{g}(\mu))^T \Sigma (\nabla\mathbf{g}(\mu)) > 0$, then $\mathbf{g}(\mathbf{X}_n) \longrightarrow \mathbf{N}(\mathbf{g}(\mu), \tau)$ in distribution. Lemma 1 is related to the the multivariate Delta method.*

Here, with the help of a theorem well known as the multivariate Delta method, we find the asymptotic distribution of $R^{|a,b}$. As $n \longrightarrow \infty$ and $m \longrightarrow \infty$; then,

$$\begin{cases} \frac{1}{\sigma_1}\left(\widehat{R}^{|a,b} - R^{|a,b}\right) \longrightarrow N(0,1)|_{a=b}, \\ \frac{1}{\sigma_2}\left(\widehat{R}^{|a,b} - R^{|a,b}\right) \longrightarrow N(0,1)||_{a<b}, \\ \frac{1}{\sigma_3}\left(\widehat{R}^{|a,b} - R^{|a,b}\right) \longrightarrow N(0,1)|_{a>b}, \end{cases} \tag{28}$$

where σ_1^2, σ_2^2, and σ_3^2 are obtained from the following equations. In all three cases,

$$\sigma^2(\lambda,\alpha_1,\alpha_2) = \nabla R^{|a,b}\mathbf{I}^{-1}(\lambda,\alpha_1,\alpha_2)\left(\nabla R^{|a,b}\right)^T,$$
$$\nabla R^{|a,b} = \left(\frac{\partial R^{|a,b}}{\partial\lambda}, \frac{\partial R^{|a,b}}{\partial\alpha_1}, \frac{\partial R^{|a,b}}{\partial\alpha_2}\right). \tag{29}$$

For all cases, $\mathbf{I}^{-1}(\lambda,\alpha_1,\alpha_2)$ is equal and the only difference is in $\nabla R^{|a,b}$.

In the following, this quantity has been computed for cases $a=b$, $a<b$, and $a>b$, respectively.

Case 1. $a=b$.

$$\begin{aligned}
\frac{\partial R^{|a,b}}{\partial\lambda} &= \frac{\left(\left[\alpha_2 b e^{-\lambda b}\left(1-e^{-\lambda b}\right)^{\alpha_2-1} - \alpha_2 b e^{-\lambda b}\left(1-e^{-\lambda b}\right)^{\alpha_1+\alpha_2-1}\right] \times \left\{\left[1-\left(1-e^{-\lambda b}\right)^{\alpha_1}\right]\left[1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right]\right\}\right)}{\left(\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right)^2} \\
&\quad - \frac{\left\{\alpha_1 b e^{-\lambda b}\left(1-e^{-\lambda b}\right)^{\alpha_1-1}\left[1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right] + \alpha_2 b e^{-\lambda b}\left(1-e^{-\lambda b}\right)^{\alpha_2-1}\left[1-\left(1-e^{-\lambda b}\right)^{\alpha_1}\right]\right\}}{\left(\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right)^2} \\
&\quad \times \frac{\left[1-\left(1-e^{-\lambda b}\right)^{\alpha_2} - \left(\alpha_2/\left(\alpha_1+\alpha_2\right)\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2+\alpha_1}\right)\right]}{\left(\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right)^2}, \\
\frac{\partial R^{|a,b}}{\partial\alpha_1} &= \frac{\left[\left(-\left(\alpha_2/\left(\alpha_1+\alpha_2\right)\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2+\alpha_1}\right) + \left(1-e^{-\lambda b}\right)^{\alpha_2+\alpha_1}\ln\left(1-e^{-\lambda b}\right)\left(\alpha_2/\left(\alpha_1+\alpha_2\right)\right)\right)\right]}{\left(\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right)^2} \\
&\quad \times \frac{\left[\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right] - \left[\left(1-e^{-\lambda b}\right)^{\alpha_1}\ln\left(1-e^{-\lambda b}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right]}{\left(\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right)^2} \\
&\quad \times \frac{\left[1-\left(1-e^{-\lambda b}\right)^{\alpha_2} - \left(\alpha_2/\left(\alpha_1+\alpha_2\right)\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2+\alpha_1}\right)\right]}{\left(\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right)^2}, \\
\frac{\partial R^{|a,b}}{\partial\alpha_2} &= \frac{\left[\left(1-e^{-\lambda b}\right)^{\alpha_2}\ln\left(1-e^{-\lambda b}\right) + \left(\alpha_2/\left(\alpha_1+\alpha_2\right)^2\right)\left(1-e^{-\lambda b}\right)^{\alpha_2+\alpha_1} + \left(\alpha_2/\left(\alpha_1+\alpha_2\right)^2\right)\ln\left(1-e^{-\lambda b}\right)\left(1-e^{-\lambda b}\right)^{\alpha_2+\alpha_1}\right]}{\left(\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right)^2} \\
&\quad \frac{\left[\left(1-\left(1-e^{-\lambda a}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right] - \left[\left(1-e^{-\lambda b}\right)^{\alpha_2}\ln\left(1-e^{-\lambda b}\right)\left(1-\left(1-e^{-\lambda a}\right)^{\alpha_1}\right)\right]}{\left(\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right)^2} \\
&\quad \frac{\left[1-\left(1-e^{-\lambda b}\right)^{\alpha_2} - \left(\alpha_2/\left(\alpha_1+\alpha_2\right)\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2+\alpha_1}\right)\right]}{\left(\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right)^2}.
\end{aligned} \tag{30}$$

Case 2. $a<b$.

$$
\begin{aligned}
\frac{\partial R^{|a,b}}{\partial \lambda} &= \frac{\left[\alpha_2 b e^{-\lambda b}\left(1-e^{-\lambda b}\right)^{\alpha_2-1} - \alpha_2 b e^{-\lambda b}\left(1-e^{-\lambda b}\right)^{\alpha_1+\alpha_2-1}\right]\left[\left(1-\left(1-e^{-\lambda a}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right]}{\left(\left(1-\left(1-e^{-\lambda a}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right)^2} \\
&\quad - \frac{\left[\alpha_1 a e^{-\lambda a}\left(1-e^{-\lambda a}\right)^{\alpha_1-1}\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right) + \alpha_2 b e^{-\lambda b}\left(1-e^{-\lambda b}\right)^{\alpha_2-1}\left(1-\left(1-e^{-\lambda a}\right)^{\alpha_1}\right)\right]}{\left(\left(1-\left(1-e^{-\lambda a}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right)^2} \\
&\quad \frac{\left[1-\left(1-e^{-\lambda b}\right)^{\alpha_2} - \left(\alpha_2/\left(\alpha_1+\alpha_2\right)\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2+\alpha_1}\right)\right]}{\left(\left(1-\left(1-e^{-\lambda a}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right)^2}, \\
\frac{\partial R^{|a,b}}{\partial \alpha_1} &= \frac{\left[\left(-\left(\alpha_2/\left(\alpha_1+\alpha_2\right)\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2+\alpha_1}\right) + \left(1-e^{-\lambda b}\right)^{\alpha_2+\alpha_1}\ln\left(1-e^{-\lambda b}\right)\left(\alpha_2/\left(\alpha_1+\alpha_2\right)\right)\right)\right]}{\left(\left(1-\left(1-e^{-\lambda a}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right)^2} \\
&\quad \frac{\left[\left(1-\left(1-e^{-\lambda a}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right] - \left[\left(1-e^{-\lambda a}\right)^{\alpha_1}\ln\left(1-e^{-\lambda a}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right]}{\left(\left(1-\left(1-e^{-\lambda a}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right)^2} \\
&\quad \frac{\left[1-\left(1-e^{-\lambda b}\right)^{\alpha_2} - \left(\alpha_2/\left(\alpha_1+\alpha_2\right)\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2+\alpha_1}\right)\right]}{\left(\left(1-\left(1-e^{-\lambda a}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right)^2}, \\
\frac{\partial R^{|a,b}}{\partial \alpha_2} &= \frac{\left[\left(1-e^{-\lambda b}\right)^{\alpha_2}\ln\left(1-e^{-\lambda b}\right) + \left(\alpha_2/\left(\alpha_1+\alpha_2\right)^2\right)\left(1-e^{-\lambda b}\right)^{\alpha_2+\alpha_1} + \left(\alpha_2/\left(\alpha_1+\alpha_2\right)^2\right)\ln\left(1-e^{-\lambda b}\right)\left(1-e^{-\lambda b}\right)^{\alpha_2+\alpha_1}\right]}{\left(\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right)^2} \\
&\quad \times \frac{\left(\left[\left(1-\left(1-e^{-\lambda a}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right] - \left[\left(1-e^{-\lambda b}\right)^{\alpha_2}\ln\left(1-e^{-\lambda b}\right)\left(1-\left(1-e^{-\lambda a}\right)^{\alpha_1}\right)\right]\right)}{\left(\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right)^2} \\
&\quad \times \frac{\left[1-\left(1-e^{-\lambda b}\right)^{\alpha_2} - \left(\alpha_2/\alpha_{1+}\alpha_2\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2+\alpha_1}\right)\right]}{\left(\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right)^2} \\
&\quad \times \frac{\left[1-\left(1-e^{-\lambda b}\right)^{\alpha_2} - \left(\alpha_2/\alpha_{1+}\alpha_2\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2+\alpha_1}\right)\right]}{\left(\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right)^2}.
\end{aligned}
\tag{31}
$$

Case 3. $a > b$.

$$\frac{\partial R^{|a,b}}{\partial \lambda} = \frac{\left[\alpha_1 a e^{-\lambda a}\left(1-e^{-\lambda a}\right)^{\alpha_1+\alpha_2-1} - \alpha_2 b e^{-\lambda b}\left(1-e^{-\lambda b}\right)^{\alpha_2-1}\left(1-\left(1-e^{-\lambda a}\right)^{\alpha_1}\right) + \alpha_1 a e^{-\lambda a}\left(1-e^{-\lambda a}\right)^{\alpha_1-1}\left(1-e^{-\lambda b}\right)^{\alpha_2}\right]}{\left(\left(1-\left(1-e^{-\lambda a}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right)^2}$$

$$\frac{\left[\left(1-\left(1-e^{-\lambda a}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right] - \left[\alpha_1 a e^{-\lambda a}\left(1-e^{-\lambda a}\right)^{\alpha_1-1}\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right) + \alpha_2 b e^{-\lambda b}\left(1-e^{-\lambda b}\right)^{\alpha_2-1}\left(1-\left(1-e^{-\lambda a}\right)^{\alpha_1}\right)\right]}{\left(\left(1-\left(1-e^{-\lambda a}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right)^2}$$

$$\frac{\left[\left(\alpha_1/(\alpha_1+\alpha_2)\right)\left(1-\left(1-e^{-\lambda a}\right)^{\alpha_1+\alpha_2}\right) - \left(\left(1-e^{-\lambda b}\right)^{\alpha_2}\left(1-\left(1-e^{-\lambda a}\right)^{\alpha_1}\right)\right)\right]}{\left(\left(1-\left(1-e^{-\lambda a}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right)^2},$$

$$\frac{\partial R^{|a,b}}{\partial \alpha_1} = \frac{\left[\left(\alpha_2/(\alpha_1+\alpha_2)^2\right)\left(1-\left(1-e^{-\lambda a}\right)^{\alpha_1+\alpha_2}\right) + \left(\alpha_1/(\alpha_1+\alpha_2)\right)\left(1-e^{-\lambda a}\right)^{\alpha_1+\alpha_2}\ln\left(1-e^{-\lambda a}\right) - \left(\left(1-e^{-\lambda a}\right)^{\alpha_1}\ln\left(1-e^{-\lambda a}\right)\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right]}{\left(\left(1-\left(1-e^{-\lambda a}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right)^2}$$

$$\frac{\left[\left(1-\left(1-e^{-\lambda a}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right] - \left[\left(1-e^{-\lambda a}\right)^{\alpha_1}\ln\left(1-e^{-\lambda a}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right]}{\left(\left(1-\left(1-e^{-\lambda a}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right)^2}$$

$$\frac{\left[\left(\alpha_1/(\alpha_1+\alpha_2)\right)\left(1-\left(1-e^{-\lambda a}\right)^{\alpha_1+\alpha_2}\right) - \left(\left(1-e^{-\lambda b}\right)^{\alpha_2}\left(1-\left(1-e^{-\lambda a}\right)^{\alpha_1}\right)\right)\right]}{\left(\left(1-\left(1-e^{-\lambda a}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right)^2},$$

$$\frac{\partial R^{|a,b}}{\partial \alpha_2} = \frac{\left[-\left(\alpha_1/(\alpha_1+\alpha_2)^2\right)\left(1-\left(1-e^{-\lambda a}\right)^{\alpha_1+\alpha_2}\right) + \left(\alpha_1/(\alpha_1+\alpha_2)\right)\left(1-e^{-\lambda a}\right)^{\alpha_1+\alpha_2}\ln\left(1-e^{-\lambda a}\right) - \left(\left(1-e^{-\lambda b}\right)^{\alpha_2}\ln\left(1-e^{-\lambda b}\right)\left(1-\left(1-e^{-\lambda a}\right)^{\alpha_1}\right)\right)\right]}{\left(\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right)^2}$$

$$\frac{\left[\left(1-\left(1-e^{-\lambda a}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right] - \left[\left(1-e^{-\lambda b}\right)^{\alpha_2}\ln\left(1-e^{-\lambda b}\right)\left(1-\left(1-e^{-\lambda a}\right)^{\alpha_1}\right)\right]}{\left(\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right)^2}$$

$$\frac{\left[1-\left(1-e^{-\lambda b}\right)^{\alpha_2} - \left(\alpha_1/(\alpha_1+\alpha_2)\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2+\alpha_1}\right)\right]}{\left(\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_1}\right)\left(1-\left(1-e^{-\lambda b}\right)^{\alpha_2}\right)\right)^2}. \tag{32}$$

Equation (28) can be used for finding the confidence interval of $R^{|a,b}$, however, by using of estimates of asymptotic variance. From this theorem, the $(1-\alpha)\times 100$ percentage confidence interval of $R^{|a,b}$ is given by

$$\begin{cases} R^{|a,b} \in \left(\widehat{R}^{|a,b} - z_{1-(\alpha/2)}\widehat{\sigma}_1, \widehat{R}^{|a,b} + z_{1-(\alpha/2)}\widehat{\sigma}_1\right)|_{a=b}, \\ R^{|a,b} \in \left(\widehat{R}^{|a,b} - z_{1-(\alpha/2)}\widehat{\sigma}_2, \widehat{R}^{|a,b} + z_{1-(\alpha/2)}\widehat{\sigma}_2\right)|_{a<b}, \\ R^{|a,b} \in \left(\widehat{R}^{|a,b} - z_{1-(\alpha/2)}\widehat{\sigma}_3, \widehat{R}^{|a,b} + z_{1-(\alpha/2)}\widehat{\sigma}_3\right)|_{a>b}. \end{cases} \tag{33}$$

In the abovementioned equations, $\widehat{\sigma}_i^2\ i=1,2,3$ are similar to $\widehat{\sigma}_i^2\ i=1,2,3$ in equation (28) with substitution $\widehat{\lambda}$, $\widehat{\alpha}_1$, and $\widehat{\alpha}_2$ instead of λ, α_1, and α_2.

3. Bayesian Method for Estimation

In this section, we provide a Bayes estimator for $R^{|a,b}$. All priors for α_1, α_2, and λ are considered Gamma distribution. We more exactly denoted

$$\pi(\alpha_1 \mid \xi_1, d_1) = \frac{d_1^{\xi_1}}{\Gamma(\xi_1)}\alpha_1^{\xi_1-1}e^{-d_1\alpha_1},$$

$$\pi(\alpha_2 \mid \xi_2, d_2) = \frac{d_2^{\xi_2}}{\Gamma(\xi_2)}\alpha_2^{\xi_2-1}e^{-d_2\alpha_2}, \tag{34}$$

$$\pi(\lambda \mid \xi_3, d_3) = \frac{d_3^{\xi_3}}{\Gamma(\xi_3)}\lambda^{\xi_3-1}e^{-d_3\lambda}.$$

Now, the joint posterior density of θ is given by

$$\begin{aligned} L(\boldsymbol{\theta} \mid x, y) &\propto \alpha_1^{m+\xi_1-1}\alpha_2^{n+\xi_2-1}\lambda^{m+n+\xi_3-1} \\ &\times e^{-\lambda\left(\sum_{i=1}^{m} x_i + \sum_{j=1}^{n} y_j + d_3\right) - d_1\alpha_1 - d_2\alpha_2} \\ &\times \left[\prod_{i=1}^{m}\left(1-e^{-\lambda x_i}\right)\right]^{\alpha_1-1}\left[\prod_{j=1}^{n}\left(1-e^{-\lambda y_j}\right)\right]^{\alpha_2-1}. \end{aligned} \tag{35}$$

Equation (35) is complicated, and it does not belong to a known distrinution. Therefore, we use Gibs sampler to

Table 1: Results for $R^{|2,2} = 0.50061$ and some values of sample sizes n and m.

n	15	25	100	40	25	30	20
m	15	25	100	10	45	60	80
$\widehat{R}^{|a,b}$	0.500656	0.5006194	0.5006128	0.500839	0.500552	0.50056	0.500506
Bias	−0.00117	-6×10^{-5}	−0.0007875	0.000149	−0.00067	0.00082	0.000148
MSE	$1.37E-06$	3.6×10^{-9}	6.2×10^{-7}	$2.2E-08$	4.4×10^{-7}	6.7×10^{-7}	2×10^{-8}
CP	0.00472	0.0033517	0.0016044	0.004563	0.002777	0.002506	0.002723
LCI	0.924	0.92	0.9478	0.9254	0.936	0.9439	0.9577

Table 2: Results for $R^{|1,3} = 0.5$ and different values of sample sizes n and m.

n	15	25	100	15	25	100
m	15	25	100	15	25	100
$\widehat{R}^{|a,b}$	0.500656	0.5006194	0.5006128	0.500656	0.500619	0.5006128
Bias	−0.00117	-6×10^{-5}	−0.0007875	−0.00117	-6×10^{-5}	−0.0007875
MSE	1.4×10^{-6}	3.6×10^{-9}	6.2×10^{-7}	1.4×10^{-6}	3.6×10^{-9}	6.20×10^{-7}
CP	0.00472	0.0033518	0.00160437	0.00472	0.0033518	0.00160437
LCI	0.924	0.92	0.9478	0.924	0.92	0.9478

generate samples from (35). By (35), the full posterior density functions are

$$L(\alpha_1 \mid \alpha_2, \lambda, \mathbf{x}, \mathbf{y}) \propto \alpha_1^{m+\xi_1-1} e^{-\alpha_1\left(d_1+\sum_{i=1}^{m}\ln\left(1-e^{-\lambda x_i}\right)\right)},$$
$$L(\alpha_2 \mid \alpha_1, \lambda, \mathbf{x}, \mathbf{y}) \propto \alpha_2^{n+\xi_2-1} e^{-\alpha_2\left(d_2+\sum_{j=1}^{n}\ln\left(1-e^{-\lambda y_j}\right)\right)}, \tag{36}$$

$$L(\lambda \mid \alpha_1, \alpha_2, \mathbf{x}, \mathbf{y}) \propto \lambda^{m+n+\xi_3-1} e^{-\lambda\left(\sum_{i=1}^{m}x_i+\sum_{j=1}^{n}y_j+d_3\right)} \times\left[\prod_{i=1}^{m}\left(1-e^{-\lambda x_i}\right)\right]^{\alpha_1-1}\left[\prod_{j=1}^{n}\left(1-e^{-\lambda y_j}\right)\right]^{\alpha_2-1}. \tag{37}$$

It is clear that the posteriors of α_1 and α_2 are Gamma distribution while (37) does not have a known distribution. Henceforth, we use Metropolis–Hastigs (M–H) algorithm to generate data from (37). The proposal distribution for M–H algorithm is considered Gamma with the shape parameter $m+n+\xi_3$ and scale parameter $\sum_{i=1}^{m}x_i+\sum_{j=1}^{n}y_j+d_3$.

4. Simulation Study

In this section, we conduct a simulation study in order to survey quality and efficiency of the introduced model and its estimator. All results are the mean of 10000 iteration. To put it more clearly, note that we have iterated our simulation 10000 times. In the i^{th} iteration, two random samples with size n and m are generated and $\widehat{R}^{|a,b}$ is computed. The values of $\widehat{R}^{|a,b}$ demonstrated in the tables are the mean of these 10000 computed estimates as follows:

$$\widehat{R}^{|a,b} = \frac{\sum_{i=1}^{10000}\widehat{R}_i^{|a,b}}{10000}. \tag{38}$$

Four criteria containing bias, Mean Square Error (MSE), Coverage Probability (CP), and Length of Confidence Interval (LCI) are used in order to investigate the effectiveness and potentialities of the method. The results are demonstrated in Table 1 for values of parameters $\alpha_1 = 2$, $\alpha_2 = 1.6$, $\lambda = 2$, $a = 2$, and $b = 2$. Also, the results are represented in Table 2 for values of parameters $\alpha_1 = 3$, $\alpha_2 = 3$, $\lambda = 1.5$, $a = 1$, and $b = 3$.

As these tables show two criteria MSE and bias are very small; therefore, our estimation method is appropriate. Also, values of CP and LCI as cover of probability and length of confidence interval for $R^{|2,2}$ represent the same findings.

5. Application

In this section, two real datasets reported by Lawless [9] (data A) and Linhardt and Zucchini [10] (data B), respectively, are analyzed. We fit the GE distribution to the two datasets separately. The first dataset arose in tests on endurance of deep groove ball bearings and is the number of million revolutions before failure for each of the 23 ball bearings in the life test. The other dataset denotes the failure times of the air-conditioning system of an airplane (in hours).

Gupta and Kundu [7] and Gupta and Kundu [11] studied the validity of GE distribution for these two datasets, respectively. In Table 3, the Kolmogorov–Smirnov distance and its corresponding P value are provided for these data which confirm that the generalized exponential model fits quite well to both the datasets where data I: 17.88, 28.92, 33, 41.52, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56, 67.8, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, and 173.40 and data II: 23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, and 52, 95.

Table 3: Results of goodness-of-fit tests for fitting GE distribution to two datasets.

Data	K–S	P value
I	0.1058	0.9592
II	0.1744	0.2926

Table 4: $\widehat{R}^{|a,b}$ and its corresponding C.I for real datasets.

$\widehat{R}^{\|1,3} = 0.5371$	$\widehat{R}^{\|2,2} = 0.5662$	$\widehat{R}^{\|1,3} = 0.5126$	$\widehat{R}^{\|0.1,0.4} = 0.6843$
(0.5125, 0.5616)	(0.5231, 0.609)	(0.5037, 0.5215)	(0.5856, 0.7828)

Using the results previously presented in Section 2, we can obtain the MLE of parameters α_1, α_2, and λ. Our computations show that $\widehat{\alpha}_1 = 2.8082$, $\widehat{\alpha}_2 = 1.0067$, and $\widehat{\lambda} = 0.5036$. For these estimated parameters, $\widehat{R}^{|a,b}$ and its corresponding confidence interval have been computed for some values of a and b, as in Table 4.

6. Conclusions

In this paper, a new extension of the stress-strength model is defined, studied, and applied under many particular cases. The novel stress-strength model is applied under the generalized exponential distribution. The maximum likelihood estimator, asymptotic distribution, and Bayesian estimation are obtained with details. A simulation study along with analysis of two real datasets is also performed for illustrative purposes. We hope that the proposed method would enable engineers and system designers to design better products.

Data Availability

All data are available on request.

Conflicts of Interest

The authors have no conflicts of interest.

References

[1] S. Kotz, Y. Lumelskii, and M. Pensky, *The Stress-Strength Model and Its Generalization: Theory and Applications*, World Scientific, Singapore, 2003.

[2] L. Ventura and W. Racugno, "Recent advances on Bayesian inference for $P(X<Y)$," *Bayesian analysis*, vol. 6, no. 2, pp. 1–18, 2011.

[3] S. Rezaei, R. Tahmasbi, and M. Mahmoodi, "Estimation of for generalized Pareto distribution," *Journal of Statistical Planning and Inference*, vol. 140, no. 2, pp. 480–494, 2010.

[4] M. Rasekhi, M. M. Saber, and H. M. Yousof, "Bayesian and classical inference of reliability in multicomponent stress-strength under the generalized logistic model," *Communications in Statistics—Theory and Methods*, vol. 50, no. 21, pp. 5114–5125, 2020.

[5] M. M. Saber and H. M. Yousof, "Bayesian and classical inference for generalized stress-strength parameter under generalized logistic distribution," *Statistics, Optimization & Information Computing*, 2021.

[6] M. M. Saber and K. Khorshidian, "Introduction to reliability for conditional stress-strength parameter," *Journal of Sciences*, 2021.

[7] R. D. Gupta and D. Kundu, "Generalized exponential distribution: different method of estimations," *Journal of Statistical Computation and Simulation*, vol. 69, no. 4, pp. 315–337, 2001.

[8] G. S. Rao, "Estimation of reliability in multicomponent stress-strength based on generalized exponential distribution," *Revista Colombiana de Estadística*, vol. 35, no. 1, pp. 67–76, 2012.

[9] J. F. Lawless, *Statistical Models and Methods for Lifetime Data*, Wiley, New York, NY, USA, 1982.

[10] H. Linhart and W. Zucchini, *Model Selection*, John Wiley & Sons, Hoboken, NJ, USA, 1986.

[11] R. D. Gupta and D. Kundu, "Discriminating between Weibull and generalized exponential distributions," *Computational Statistics & Data Analysis*, vol. 43, no. 2, pp. 179–196, 2003.

[12] S. G. Fletcher and K. Ponnambalam, "Estimation of reservoir yield and storage distribution using moments analysis," *Journal of Hydrology*, vol. 182, no. 1–4, pp. 259–275, 1996.

[13] A. Ganji, K. Ponnambalam, D. Khalili, and M. Karamouz, "Grain yield reliability analysis with crop water demand uncertainty," *Stochastic Environmental Research and Risk Assessment*, vol. 20, no. 4, pp. 259–277, 2006.