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Research Article

Some Improved Classes of Estimators in Stratified Sampling Using Bivariate Auxiliary Information

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This manuscript considers some improved combined and separate classes of estimators of population mean using bivariate auxiliary information under stratified simple random sampling. The expressions of bias and mean square error of the proposed classes of estimators are determined to the first order of approximation. It is exhibited that under some particular conditions, the proposed classes of estimators dominate the existing prominent estimators. The theoretical findings are supported by a simulation study performed over a hypothetically generated population.

1. Introduction

In sampling theory, the appropriate utilization of auxiliary information plays a leading role to improve the efficiency of the estimators. This information may be utilized either at the design phase (sampling design) or at the estimation phase or in both phases. It is very popular when auxiliary information is considered at the estimation phase, the ratio, product, regression, and exponential type estimators are mostly the preferred methods in different dimensions. Shabbir et al. [1] examined the performance of ratio-exponential log type class of estimators using two auxiliary variables. Shahzad et al. [2] introduced a novel family of variance estimators based on L-moments and calibration approach under stratified simple random sampling (*SSRS*), whereas Shahzad et al. [3] suggested L-Moments and calibration-based variance estimators under double *SSRS* and discussed an application of COVID-19 pandemic. Shahzad et al. [4] considered the estimation of coefficient of variation using L-moments and calibration approach for nonsensitive and sensitive variables, whereas Shahzad et al. [5] developed variance estimation based on L-moments and auxiliary information. The estimation of population mean is a widely discussed approach in sample surveys and many renowned authors have utilized these auxiliary pieces of information at estimation stage and suggested various modified estimators to date. Especially, under the availability of multi-auxiliary information, the literature contains different kinds of ratio, product, regression, and exponential type estimators.In *SSRS*, the utilization of auxiliary information at the estimation stage has been discussed by many authors to enhance the efficiency of the estimators. In presence of univariate auxiliary information, Hansen et al. [6]; Kadilar and Cingi [7]; Shabbir and Gupta [8]; Singh and Vishwakarma [9]; Solanki and Singh [10]; Bhushan et al. [11]; etc., suggested various modified estimators of population mean whereas, in presence of multi-auxiliary information, Koyuncu and Kadilar [12] suggested a family of estimators of population mean based on *SSRS*. Tailor et al. [13] envisaged ratio-cum-product estimator of population mean in *SSRS*. Tailor and Chouhan [14] addressed ratio-cum-product type exponential estimator of finite population mean. Following Upadhyaya et al. [15]; Singh et al. [16] considered a class of ratio-cum-product estimators using information on two auxiliary variables in *SSRS*. Along the lines of Singh et al. [17]; Lone et al. [18] introduced a generalized ratio-cum-product type exponential estimator in *SSRS*, whereas Lone et al. [19] suggested efficient separate class of estimators of population mean. Following Singh and Singh [20]; Muneer et al. [21] suggested a class of combined estimators in *SSRS*. Recently, Muneer et al. [22] introduced a chain ratio exponential family of estimators based on *SSRS*. In the present study, we propose some improved classes of estimators for the estimation of population mean by employing bivariate auxiliary information under *SSRS*.Consider a finite population $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_N)$ based on size N units with study variable y and two auxiliary variables x and z, respectively, associated with each unit $\kappa_i, i = 1, 2, \ldots, N$ of the population. Let the population be divided into L disjoint strata with the stratum h comprises of $N_h, h = 1, 2, \ldots, L$ units. Let a simple random sample of size n_h be quantified without replacement from the stratum h such that $\sum_{h=1}^{L} n_h = n$. Let the observed values of y, x and z on the i^{th} unit of the stratum h be denoted by $(y_{h_i}, x_{h_i}, z_{h_i})$, $i = 1, 2, \ldots, N_h$. Let $\overline{y}_{st} = \sum_{h=1}^{L} W_h \overline{y}_h$, $\overline{x}_{st} = \sum_{h=1}^{L} W_h \overline{x}_h$ and $\overline{z}_{st} = \sum_{h=1}^{L} W_h \overline{z}_h$, respectively, be the sample means corresponding to the population means $\overline{Y} = \overline{Y}_{st} = \sum_{h=1}^{L} W_h \overline{Y}_h$, $\overline{X} = \overline{X}_{st} = \sum_{h=1}^{L} W_h \overline{X}_h$ and $\overline{Z} = \overline{Z}_{st} = \sum_{h=1}^{L} W_h \overline{Z}_h$ of variables y, x and z, where $W_h = N_h/N$ is the weight in the stratum h. Let $\overline{y}_h = \sum_{i=1}^{n_h} y_{h_i}/n_h$, $\overline{x}_h = \sum_{i=1}^{n_h} x_{h_i}/n_h$ and $\overline{z}_h = \sum_{i=1}^{n_h} z_{h_i}/n_h$be the sample means corresponding to the population means $\overline{Y}_h = \sum_{i=1}^{N_h} Y_{h_i}/N_h$, $\overline{X}_h = \sum_{i=1}^{N_h} X_{h_i}/N_h$ and $\overline{Z}_h = \sum_{i=1}^{N_h} Z_{h_i}/N_h$ of variables y, x and z in the stratum h. Let $s_{y_h}^2 = \sum_{h=1}^{L} (y_{hi} - \overline{y}_h)^2/(n_h - 1)$, $s_{x_h}^2 = \sum_{h=1}^{L} (x_{hi} - \overline{x}_h)^2/(n_h - 1)$, $s_{z_h}^2 = \sum_{h=1}^{L} (z_{hi} - \overline{z}_h)^2/(n_h - 1)$, $s_{xy_h} = \sum_{h=1}^{L}((x_{hi} - \overline{x}_h))(y_{hi} - \overline{y}_h))/(n_h - 1)$, $s_{zy_h} = \sum_{h=1}^{L}((z_{hi} - \overline{z}_h))(y_{hi} - \overline{y}_h))/(n_h - 1)$, $s_{xz_h} = \sum_{h=1}^{L}((x_{hi} - \overline{x}_h))(z_{hi} - \overline{z}_h))/(n_h - 1)$, respectively, be the sample variances and covariances corresponding to the population variances and covariances $S_{y_h}^2 = \sum_{h=1}^{L} (y_{hi} - \overline{Y}_h)^2/(N_h - 1)$, $S_{x_h}^2 = \sum_{h=1}^{L} (x_{hi} - \overline{X}_h)^2/(N_h - 1)$, $S_{z_h}^2 = \sum_{h=1}^{L} (z_{hi} - \overline{Z}_h)^2/(N_h - 1)$, $S_{xy_h} = \sum_{h=1}^{L} (x_{hi} - \overline{X}_h)(N_h - 1)/(N_h - 1)$, $S_{zy_h} = \sum_{h=1}^{L}((z_{hi} - \overline{Z}_h))(y_{hi} - \overline{Y}_h))/(N_h - 1)$, $S_{xz_h} = \sum_{h=1}^{L}((x_{hi} - \overline{X}_h))(z_{hi} - \overline{Z}_h))/(N_h - 1)$ in the stratum h.

To derive the bias and mean square error (*MSE*) of different combined estimators, the following notations will be used throughout the paper.

$\overline{y}_{st} = \overline{Y} + \varepsilon_0, \overline{x}_{st} = \overline{X} + \varepsilon_1, \overline{z}_{st} = \overline{Z} + \varepsilon_2$, such that $E(\varepsilon_0) = E(\varepsilon_1) = E(\varepsilon_2) = 0$ and

$$V_{rst} = \sum_{h=1}^{L} W_h^{r+s+t} E\left[(\overline{y}_h - \overline{Y}_h)^r (\overline{x}_h - \overline{X}_h)^s (\overline{z}_h - \overline{Z}_h)^t\right]. \quad (1)$$

Following (1), we can write $E(\varepsilon_0^2) = \sum_{h=1}^{L} W_h^2 \gamma_h S_{y_h}^2 = V_{200}$; $E(\varepsilon_1^2) = \sum_{h=1}^{L} W_h^2 \gamma_h S_{x_h}^2 = V_{020}$; $E(\varepsilon_2^2) = \sum_{h=1}^{L} W_h^2 \gamma_h S_{z_h}^2 = V_{002}$; $E(\varepsilon_0, \varepsilon_1) = \sum_{h=1}^{L} W_h^2 \gamma_h \rho_{xy_h} S_{x_h} S_{y_h} = V_{110}$; $E(\varepsilon_1, \varepsilon_2) = \sum h = 1 L W_h^2 \gamma_h \rho_{xz_h} S_{x_h} S_{z_h} = V_{011}$ and $E(\varepsilon_0, \varepsilon_2) = \sum_{h=1}^{L} W_h^2 \gamma_h \rho_{yz_h} S_{y_h} S_{z_h} = V_{101}$, where $\gamma = 1/n_h$ and ρ_{xy_h}, ρ_{xz_h} and ρ_{yz_h} be the population coefficient of correlation with their respective subscripts in stratum h.

Again, to determine the bias and *MSE* of the separate estimators, the following notations will be used throughout the paper.

$\overline{y}_h = \overline{Y}_h + \varepsilon_{0h}$, $\overline{x}_h = \overline{X}_h + \varepsilon_{1h}$, $\overline{z}_h = \overline{Z}_h + \varepsilon_{2h}$ such that $E(\varepsilon_{0h}) = E(\varepsilon_{1h}) = E(\varepsilon_{2h}) = 0$ and $E(\varepsilon_{0h}^2) = \gamma_h S_{y_h}^2 = U_{200}$; $E(\varepsilon_{1h}^2) = \gamma_h S_{x_h}^2 = U_{020}$; $E(\varepsilon_{2h}^2) = \gamma_h S_{z_h}^2 = U_{002}$; $E(\varepsilon_{0h}, \varepsilon_{1h}) = \gamma_h \rho_{xy_h} S_{x_h} S_{y_h} = U_{110}$; $E(\varepsilon_{1h}, \varepsilon_{2h}) = \gamma_h \rho_{xz_h} S_{x_h} S_{z_h} = U_{011}$ and $E(\varepsilon_{0h}, \varepsilon_{2h}) = \gamma_h \rho_{yz_h} S_{y_h} S_{z_h} = U_{101}$.

The aim of the present paper is to suggest some improved combined and separate classes of estimators in the presence of bivariate auxiliary information under *SSRS*. The remainder of the paper is drafted in the following sections. Section 2 deals with the existing combined and separate estimators, whereas Section 3 considers the proposed combined and separate classes of estimators along with their properties. The theoretical comparison of the proposed combined and separate classes of estimators with the existing combined and separate estimators is given in Section 4. The credibility of the theoretical findings is furnished with a simulation study in Section 5. Finally, a conclusion of this study is drawn in Section 6.

2. Existing Estimators

2.1. Combined Estimators. The conventional combined mean estimator under *SSRS* is given by

$$y_m^c = \overline{y}_{st}. \quad (2)$$

On the lines of Singh [23]; one may consider the classical combined ratio estimator of population mean $\overline{Y}$ using bivariate auxiliary information under *SSRS* as

$$\overline{y}_r^c = \overline{y}_{st}\left(\frac{\overline{X}}{\overline{x}_{st}}\right)\left(\frac{\overline{Z}}{\overline{z}_{st}}\right). \quad (3)$$

The classical combined regression estimator of population mean $\overline{Y}$ based on bivariate auxiliary information under *SSRS* is given by

$$\overline{y}_l^c = \overline{y}_{st} + \beta_1(\overline{X} - \overline{x}_{st}) + \beta_2(\overline{Z} - \overline{z}_{st}), \quad (4)$$

where β_1 and β_2 are the regression coefficients of y on x and z, respectively.

Following Olkin [24]; the combined ratio type estimator using bivariate auxiliary information under *SSRS* is given by

$$\overline{y}_o^c = \overline{y}_{st}\left\{w\left(\frac{\overline{X}}{\overline{x}_{st}}\right) + (1 - w)\left(\frac{\overline{Z}}{\overline{z}_{st}}\right)\right\}, \quad (5)$$

where w is a duly opted scalar.

Along the lines of Abu–Dayyeh et al. [25]; some combined classes of ratio type estimators using bivariate auxiliary information under *SSRS* are given by

$$\tilde{y}_a^c = \overline{y}_{st}\left(\frac{\overline{X}}{\overline{x}_{st}}\right)^{a_1}\left(\frac{\overline{Z}}{\overline{z}_{st}}\right)^{a_2}, \quad (6)$$

$$\tilde{y}_w^c = \overline{y}_{st}\left\{w_1\left(\frac{\overline{X}}{\overline{x}_{st}}\right)^{a_1} + w_2\left(\frac{\overline{Z}}{\overline{z}_{st}}\right)^{a_2}\right\}, \quad (7)$$

where a_1, a_2, w_1, and w_2 are duly opted scalars and $w_1 + w_2 = 1$.

Following Kadilar and Cingi [7], one may suggest some combined ratio-cum-product estimators based on bivariate auxiliary information under *SSRS* as

$$\overline{y}^c_{kc_1} = \overline{y}_{st}\frac{\sum_{h=1}^{L} W_h\left(\overline{X}_h + C_{x_h}\right)\left(\overline{Z}_h + C_{z_h}\right)}{\sum_{h=1}^{L} W_h\left(\overline{x}_h + C_{x_h}\right)\left(\overline{z}_h + C_{z_h}\right)}, \quad (8)$$

$$\overline{y}^c_{kc_2} = \overline{y}_{st}\frac{\sum_{h=1}^{L} W_h\left(\overline{X}_h + \beta_2(x_h)\right)\left(\overline{Z}_h + \beta_2(z_h)\right)}{\sum_{h=1}^{L} W_h\left(\overline{x}_h + \beta_2(x_h)\right)\left(\overline{z}_h + \beta_2(z_h)\right)}, \quad (9)$$

$$\overline{y}^c_{kc_3} = \overline{y}_{st}\frac{\sum_{h=1}^{L} W_h\left(\overline{X}_h\beta_2(x_h) + C_{x_h}\right)\left(\overline{Z}_h\beta_2(z_h) + C_{z_h}\right)}{\sum_{h=1}^{L} W_h\left(\overline{x}_h\beta_2(x_h) + C_{x_h}\right)\left(\overline{z}_h\beta_2(z_h) + C_{z_h}\right)}, \quad (10)$$

$\overline{y}^c_{kc_4} = \overline{y}_{st} \sum_{h=1}^{L} W_h \quad (\overline{X}_h C_{x_h} + \beta_2(x_h))(\overline{Z}_h C_{z_h} + \beta_2(z_h))$ $\sum_{h=1}^{L} W_h \quad (\overline{x}_h C_{x_h} + \beta_2(x_h))(\overline{z}_h C_{z_h} + \beta_2(z_h))$, (11) where $\beta_2(x_h)$ and $\beta_2(z_h)$ are the coefficient of kurtosis of variables x and z, respectively, in stratum h.

Following Upadhyaya et al. [15]; Singh et al. [16] considered a combined class of ratio-cum-product estimators using bivariate auxiliary information in *SSRS* as

$$\overline{y}^c_s = \eta_1\overline{y}_{st} + \eta_2\overline{y}_{st}\left(\frac{\overline{X}}{\overline{x}_{st}}\right)^{\alpha}\left(\frac{\overline{z}_{st}}{\overline{Z}}\right)^{\beta}, \quad (12)$$

where η_1, η_2 are duly opted scalars and α, β are scalars taking real values.

Motivated by Khoshnevisan et al. [26]; Koyuncu and Kadilar [12] suggested a general family of combined estimators for population mean $\overline{Y}$ using bivariate auxiliary information under *SSRS* as

$$\overline{y}^c_k = \overline{y}_{st}\left\{\frac{a\overline{X} + b}{\lambda_1(a\overline{x}_{st} + b) + (1 - \lambda_1)(a\overline{X} + b)}\right\}^{g_1}\left\{\frac{c\overline{Z} + d}{\lambda_2(c\overline{z}_{st} + d) + (1 - \lambda_2)(c\overline{Z} + d)}\right\}^{g_2}, \quad (13)$$

where λ_1, λ_2, g_1, and g_2 are prescribed scalars, whereas $a(\neq 0)$, b and $c(\neq 0)$, d are either real numbers or functions of the known parameters of auxiliary variables x and z, respectively.

Along the lines of Singh et al. [17]; Tailor and Chouhan [14] suggested a combined ratio-cum-product type exponential estimator for population mean using bivariate auxiliary information under *SSRS* as

$$\overline{y}^c_{tc} = k\overline{y}_{st}\exp\left(\frac{\overline{X} - \overline{x}_{st}}{\overline{X} + \overline{x}_{st}}\right)\exp\left(\frac{\overline{z}_{st} - \overline{Z}}{\overline{z}_{st} + \overline{Z}}\right), \quad (14)$$

where k is a duly opted scalar.

Following Singh et al. [17], Lone et al. [18] suggested a combined generalized ratio-cum-product type exponential estimator in *SSRS* as

$$\overline{y}^c_{l_1} = \overline{y}_{st}\exp\left\{L_1\left(\frac{\overline{X} - \overline{x}_{st}}{\overline{X} + \overline{x}_{st}}\right) + L_2\left(\frac{\overline{z}_{st} - \overline{Z}}{\overline{z}_{st} + \overline{Z}}\right)\right\}, \quad (15)$$

where L_1 and L_2 are duly opted scalars.

The combined version of Lone et al. [19] estimator for estimating population mean $\overline{Y}$ is given by

$$\overline{y}_{l_2} = \overline{y}_{st}\left(\frac{a\overline{X} + b}{a\overline{x}_{st} + b}\right)\left(\frac{c\overline{z}_{st} + d}{c\overline{Z} + d}\right). \quad (16)$$

We remark that the minimum *MSE* of Abu-Dayyeh et al. [25] type estimator $\overline{y}^c_a$, Koyuncu and Kadilar [12] estimator $\overline{y}^c_{kk}$, and Lone et al. [18, 19] estimators $\overline{y}^c_{l_i}, i = 1, 2$ are equal to the minimum *MSE* of the classical regression estimator $\overline{y}^c_l$.

Along the lines of Singh and Singh [20]; Muneer et al. [21] introduced a class of combined estimators in *SSRS* as

$$\overline{y}^c_{mu} = \{k_3\overline{y}_{st} - k_4(\overline{x}_{st} - \overline{X})\}\left[\Theta\left\{2 - \exp\left(\frac{\overline{z}_{st} - \overline{Z}}{\overline{z}_{st} + \overline{Z}}\right)\right\} + (1 - \Theta)\left\{\exp\left(\frac{\overline{Z} - \overline{z}_{st}}{\overline{Z} - \overline{z}_{st}}\right)\right\}\right], \quad (17)$$

where k_3, k_4 are duly opted scalars and Θ is a scalar assuming values 0 and 1 to design different estimators.

The combined form of Muneer et al. [22] chain ratio exponential family of estimator in *SSRS* is given by

$$\overline{y}^c_{mu_1} = \overline{y}_{st}\left(\frac{\overline{X}^*}{w_1\overline{x}^*_{st} + \left(1 - w_1\overline{X}^*\right)}\right)^{\alpha_1}\left(\frac{\overline{Z}^*}{w_2\overline{z}^*_{st} + \left(1 - w_2\overline{Z}^*\right)}\right)^{\alpha_2}\exp\left\{\frac{\alpha_3(\overline{X} - \overline{x}_{st})}{(\overline{X} + \overline{x}_{st})}\right\}\exp\left\{\frac{\alpha_4(\overline{Z} - \overline{z}_{st})}{(\overline{Z} + \overline{z}_{st})}\right\}. \quad (18)$$

On the lines of Searls [27]; an improved form of the above-combined estimator is given by

$$\overline{y}^c_{mu_2} = k_1\overline{y}_{st}\left(\frac{\overline{X}^*}{w_1\overline{x}^*_{st} + \left(1 - w_1\overline{X}^*\right)}\right)^{\alpha_1}\left(\frac{\overline{Z}^*}{w_2\overline{z}^*_{st} + \left(1 - w_2\overline{Z}^*\right)}\right)^{\alpha_2}\exp\left\{\frac{\alpha_3\left(\overline{X} - \overline{x}_{st}\right)}{\left(\overline{X} + \overline{x}_{st}\right)}\right\}\exp\left\{\frac{\alpha_4\left(\overline{Z} - \overline{z}_{st}\right)}{\left(\overline{Z} + \overline{z}_{st}\right)}\right\}, \tag{19}$$

where $\overline{X}^* = a\overline{X} + b$, $\overline{X}^* = a\overline{X} + b$, $\overline{x}^*_{st} = a\overline{x}_{st} + b$, $\overline{Z}^* = c\overline{Z} + d$, $\overline{z}^*_{st} = c\overline{z}_{st} + d$, w_1, $w_2 = (0, 1)$ and k_1 is a duly opted scalar, α_j, $j = 1, 2, 3, 4$ assumes values −1, 0, and +1 to form different new and existing estimators. Moreover, the authors have shown that more than 65 combined classes of estimators are the members of the estimators $\overline{y}^c_{mu_1}$ and $\overline{y}^c_{mu_2}$, respectively, for different values of scalars.

The bias and *MSE* of the estimators considered in this section are readily discussed in Appendix A.

2.2. Separate Estimators. The conventional separate mean estimator under *SSRS* is given by

$$\overline{y}^s_m = \sum_{h=1}^{L} W_h\overline{y}_h. \tag{20}$$

On the lines of Singh [23]; the classical separate ratio estimator of population mean $\overline{y}$ using bivariate auxiliary information under *SSRS* is defined as

$$\overline{y}^s_r = \sum_{h=1}^{L} W_h\overline{y}_h\left(\frac{\overline{X}_h}{\overline{x}_h}\right)\left(\frac{\overline{Z}_h}{\overline{z}_h}\right). \tag{21}$$

The classical separate regression estimator of population mean $\overline{Y}$ under bivariate auxiliary information using *SSRS* is

$$\overline{y}^s_l = \sum_{h=1}^{L} W_h\left\{\overline{y}_h + \beta_{1_h}\left(\overline{X}_h - \overline{x}_h\right) + \beta_{2_h}\left(\overline{Z}_h - \overline{z}_h\right)\right\}, \tag{22}$$

where β_{1_h} and β_{2_h} are the regression coefficients of y on x and z, respectively, in stratum h.

Motivated by Olkin [24]; the separate ratio estimator in *SSRS* using bivariate auxiliary information is given by

$$\overline{y}^s_o = \sum_{h=1}^{L} W_h\overline{y}_h\left\{w_h\left(\frac{\overline{X}_h}{\overline{x}_h}\right) + (1 - w_h)\left(\frac{\overline{Z}_h}{\overline{z}_h}\right)\right\}, \tag{23}$$

where w_h is a duly opted scalar in the stratum h to be determined.

Following Abu–Dayyeh et al. [25]; a separate class of ratio type estimators using bivariate auxiliary information under *SSRS* is given by

$$\overline{y}^s_a = \sum_{h=1}^{L} W_h\overline{y}_h\left(\frac{\overline{X}_h}{\overline{x}_h}\right)^{a_{1_h}}\left(\frac{\overline{Z}_h}{\overline{z}_h}\right)^{a_{2_h}}, \tag{24}$$

$$\overline{y}_{w_h} = \sum_{h=1}^{L} W_h\overline{y}_h\left\{w_{1_h}\left(\frac{\overline{X}_h}{\overline{x}_h}\right)^{a_{1_h}} + w_{2_h}\left(\frac{\overline{Z}_h}{\overline{z}_h}\right)^{a_{2_h}}\right\}, \tag{25}$$

where $a_{1_h}, a_{2_h}, w_{1_h}$ and w_{2_h} are duly opted scalars in stratum h and $w_{1_h} + w_{2_h} = 1$.

Following Kadilar and Cingi [7]; one can suggest some separate ratio-cum-product type estimators based on bivariate auxiliary information under *SSRS* as

$$\overline{y}^s_{kc_1} = \sum_{h=1}^{L} W_h\overline{y}_h\frac{\left(\overline{X}_h + C_{x_h}\right)\left(\overline{Z}_h + C_{z_h}\right)}{\left(\overline{x}_h + C_{x_h}\right)\left(\overline{z}_h + C_{z_h}\right)}, \tag{26}$$

$$\overline{y}^s_{kc_2} = \sum_{h=1}^{L} W_h\overline{y}_h\frac{\left(\overline{X}_h + \beta_2(x_h)\right)\left(\overline{Z}_h + \beta_2(z_h)\right)}{\left(\overline{x}_h + \beta_2(x_h)\right)\left(\overline{z}_h + \beta_2(z_h)\right)}, \tag{27}$$

$$\overline{y}^s_{kc_3} = \sum_{h=1}^{L} W_h\overline{y}_h\frac{\left(\overline{X}_h\beta_2(x_h) + C_{x_h}\right)\left(\overline{Z}_h\beta_2(z_h) + C_{z_h}\right)}{\left(\overline{x}_h\beta_2(x_h) + C_{x_h}\right)\left(\overline{z}_h\beta_2(z_h) + C_{z_h}\right)}, \tag{28}$$

$$\overline{y}^c_{kc_4} = \sum_{h=1}^{L} W_s\overline{y}_h\frac{\left(\overline{X}_hC_{x_h} + \beta_2(x_h)\right)\left(\overline{Z}_hC_{z_h} + \beta_2(z_h)\right)}{\left(\overline{x}_hC_{x_h} + \beta_2(x_h)\right)\left(\overline{z}_hC_{z_h} + \beta_2(z_h)\right)}. \tag{29}$$

The separate version of Singh et al. [16] estimator is given by

$$\overline{y}^s_s = \sum_{h=1}^{L} W_h\left\{\eta_{1_h}\overline{y}_h + \eta_{2_h}\overline{y}_h\left(\frac{\overline{X}_h}{\overline{x}_h}\right)^{\alpha_h}\left(\frac{\overline{z}_h}{\overline{Z}_h}\right)^{\beta_h}\right\}, \tag{30}$$

where η_{1_h}, η_{2_h} are duly opted scalars in stratum h and α_h, β_h are scalars in stratum h taking real values.

The separate version of Koyuncu and Kadilar [12] family of estimators is given by

$$\overline{y}^s_{kk} = \sum_{h=1}^{L} W_h\overline{y}_h\left\{\frac{a_h\overline{X}_h + b_h}{\lambda_{1_h}\left(a_h\overline{x}_h + b_h\right) + \left(1 - \lambda_{1_h}\right)\left(a_h\overline{X}_h + b_h\right)}\right\}^{g_1}\left\{\frac{c_h\overline{Z}_h + d_h}{\lambda_{2_h}\left(c_h\overline{z}_h + d_h\right) + \left(1 - \lambda_{2_h}\right)\left(c_h\overline{Z}_h + d_h\right)}\right\}^{g_2}, \tag{31}$$

where $\lambda_{1_h}, \lambda_{2_h}, g_1$ and g_2 are some prescribed scalars whereas $a_h(\neq 0)$, b_h and $c_h(\neq 0)$, d_h are either real numbers or functions of the known parameters of the auxiliary variables x and z, respectively, in stratum h.

The separate version of Tailor and Chouhan [14] estimator for population mean using bivariate auxiliary information under *SSRS* is defined as

$$\overline{y}^s_{tc} = \sum_{h=1}^{L} Wk_h\overline{y}_h\exp\left(\frac{\overline{X}_h - \overline{x}_h}{\overline{X}_h + \overline{x}_h}\right)\exp\left(\frac{\overline{z}_h - \overline{Z}_h}{\overline{z}_h + \overline{Z}_h}\right)_h, \tag{32}$$

where k_h is a duly opted scalar in stratum h.

On the lines of Lone et al. [18]; a generalized separate ratio-cum-product type exponential estimator in *SSRS* is defined as

$$\overline{y}_{l_1}^s = \sum_{h=1}^{L} W_h \overline{y}_h \exp\left\{L_{1_h}\left(\frac{\overline{X}_h - \overline{x}_h}{\overline{X}_h + \overline{x}_h}\right) + L_{2_h}\left(\frac{\overline{z}_h - \overline{Z}_h}{\overline{z}_h + \overline{Z}_h}\right)\right\}, \tag{33}$$

where L_{1_h} and L_{2_h} are duly opted scalars in the stratum h.

The separate version of Lone et al. [19] estimator for estimating population mean $\overline{Y}$ as

$$\overline{y}_{l_2}^s = \sum_{h=1}^{L} W_h \overline{y}_h \left(\frac{a_h \overline{X}_h + b_h}{a_h \overline{x}_h + b_h}\right)\left(\frac{c_h \overline{z}_h + d_h}{c_h \overline{Z}_h + d_h}\right). \tag{34}$$

It is to be noted that the minimum MSE of separate Abu–Dayyeh [25] type estimator $\overline{y}_a^s$, Koyuncu and Kadilar [12] estimator $\overline{y}_{kk}^s$ and Lone et al. [18, 19] estimators $\overline{y}_{l_1}^s$ & $\overline{y}_{l_2}^s$ are equal to the minimum MSE of classical separate regression estimator $\overline{y}_l^s$.The separate type of Muneer et al. [21] estimator in $SSRS$ is given by

$$\overline{y}_{mu}^s = \sum_{h=1}^{L} W_h \left\{k_{3_h}\overline{y}_h - k_{4_h}\left(\overline{x}_h - \overline{X}_h\right)\right\}\left[\Theta_h\left\{2 - \exp\left(\frac{\overline{z}_h - \overline{Z}_h}{\overline{z}_h + \overline{Z}_h}\right)\right\} + \left(1 - \Theta_h\right)\left\{\exp\left(\frac{\overline{Z}_h - \overline{z}_h}{\overline{Z}_h - \overline{z}_h}\right)\right\}\right], \tag{35}$$

where k_{3_h} and k_{4_h} are suitably chosen scalars in stratum h and Θ_h is a real constant in stratum h.

Muneer et al. [22] suggested a separate chain ratio exponential family of estimators in $SSRS$ as

$$\overline{y}_{mu_1}^s = \sum_{h=1}^{L} W_h \left[\begin{array}{c} \overline{y}_h \left\{\dfrac{\overline{X}_h^*}{w_{1_h}\overline{x}_h^* + \left(1 - w_{1_h}\right)\overline{X}_h^*}\right\}^{\alpha_{1_h}} \left\{\dfrac{\overline{Z}_h^*}{w_{2_h}\overline{z}_h^* + \left(1 - w_{2_h}\right)\overline{Z}_h^*}\right\}^{\alpha_{2_h}} \\ \times \exp\left\{\dfrac{\alpha_{3_h}\left(\overline{X}_h - \overline{x}_h\right)}{\left(\overline{X}_h + \overline{x}_h\right)}\right\} \exp\left\{\dfrac{\alpha_{4_h}\left(\overline{Z}_h - \overline{z}_h\right)}{\left(\overline{Z}_h + \overline{z}_h\right)}\right\} \end{array}\right]. \tag{36}$$

On the lines of Searls [27], the modified form of the above separate estimator is given by

$$\overline{y}_{mu_2}^s = \sum_{h=1}^{L} W_h \left[\begin{array}{c} k_{1_h}\overline{y}_h \left\{\dfrac{\overline{X}_h^*}{w_{1_h}\overline{x}_h^* + \left(1 - w_{1_h}\right)\overline{X}_h^*}\right\}^{\alpha_{1_h}} \left\{\dfrac{\overline{Z}_h^*}{w_{1_h}\overline{z}_h^* + \left(1 - w_{1_h}\right)\overline{Z}_h^*}\right\}^{\alpha_{2_h}} \\ \times \exp\left\{\dfrac{\alpha_{3_h}\left(\overline{X}_h - \overline{x}_h\right)}{\left(\overline{X}_h + \overline{x}_h\right)}\right\} \exp\left\{\dfrac{\alpha_{4_h}\left(\overline{Z}_h - \overline{z}_h\right)}{\left(\overline{Z}_h + \overline{z}_h\right)}\right\} \end{array}\right], \tag{37}$$

where$\overline{X}^* = a_h\overline{X}_h + b_h$, $\overline{x}_h^* = a_h\overline{x}_h + b_h$, $\overline{Z}^* = c_h\overline{Z}_h + d_h$, $\overline{z}_h^* = c_h\overline{z}_h + d_h$ and $w_{1_h}, w_{2_h} = (0, 1)$ and k_{1_h} is a duly opted scalar in stratum h, $\alpha_{j_h}, j = 1, 2, 3, 4$ assumes values −1, 0, and +1 in order to form different new and existing separate estimators. Furthermore, one can generate more than 65 separate classes of estimators from $\overline{y}_{mu_1}^s$ and $\overline{y}_{mu_2}^s$ for different values of scalars.The bias and MSE of the estimators considered in this section are readily discussed in the Appendix B.

3. Proposed Estimators

The objective of this paper is to suggest some improved combined and separate classes of estimators over the existing combined and separate estimators discussed in the previous section. We have extended the work of Bhushan et al. [28] for the estimation of population mean $\overline{Y}$ by incorporating bivariate auxiliary information under $SSRS$.

3.1. Combined Estimators. We propose some improved combined classes of estimators based on bivariate auxiliary information under $SSRS$ as

$$\overline{y}_{s_1}^c = \xi_1 \overline{y}_{st}\left[1 + \log\left(\frac{\overline{x}_{st}}{\overline{X}}\right)\right]^{\theta_1}\left[1 + \log\left(\frac{\overline{z}_{st}}{\overline{Z}}\right)\right]^{\delta_1}, \tag{38}$$

$$\overline{y}_{s_2}^c = \xi_2 \overline{y}_{st}\left[1 + \theta_2 \log\left(\frac{\overline{x}_{st}}{\overline{X}}\right)\right]\left[1 + \delta_2 \log\left(\frac{\overline{z}_{st}}{\overline{Z}}\right)\right], \tag{39}$$

$$\overline{y}_{s_3}^c = \xi_3 \overline{y}_{st} + \theta_3 (\overline{x}_{st} - \overline{X}) + \delta_3 (\overline{z}_{st} - \overline{Z}), \tag{40}$$

$$\overline{y}_{s_4}^c = \xi_4 \overline{y}_{st} \left(\frac{\overline{X}}{\overline{x}_{st}}\right)^{\theta_4} \left(\frac{\overline{Z}}{\overline{z}_{st}}\right)^{\delta_4}, \tag{41}$$

$$\overline{y}_{s_5}^c = \xi_5 \overline{y}_{st} \left[\frac{\overline{X}}{\overline{X} + \theta_5 (\overline{x}_{st} - \overline{X})}\right] \left[\frac{\overline{Z}}{\overline{Z} + \delta_5 (\overline{z}_{st} - \overline{Z})}\right], \tag{42}$$

where ξ_i, θ_i and δ_i, $i = 1, 2, \ldots, 5$ are duly opted scalars to be determined.

Theorem 1. *The bias of the proposed combined classes of estimators $\overline{y}_{s_i}^c, i = 1, 2, \ldots, 5$ is given by*

$$\mathrm{Bias}\left(\overline{y}_{s_1}^c\right) = \overline{Y}\left[\xi_1\left(\left\{\begin{array}{c} 1 + \left(\dfrac{\theta_1^2}{2} - \theta_1\right)V_{020} + \left(\dfrac{\delta_1^2}{2} - \delta_1\right)V_{002} + \theta_1 V_{110} \\ + \delta_1 V_{101} + \theta_1 \delta_1 V_{011} \end{array}\right\}\right) - 1\right], \tag{43}$$

$$\mathrm{Bias}\left(\overline{y}_{s_2}^c\right) = \overline{Y}\left[\xi_2\left\{1 - \frac{\theta_2}{2}V_{020} - \frac{\delta_2}{2}V_{002} + \theta_2 V_{110} + \delta_2 V_{101} + \theta_2\delta_2 V_{011}\right\} - 1\right], \tag{44}$$

$$\mathrm{Bias}\left(\overline{y}_{s_3}^c\right) = \overline{Y}(\xi_3 - 1), \tag{45}$$

$$\mathrm{Bias}\left(\overline{y}_{s_4}^c\right) = \overline{Y}\left[\xi_4\left\{\begin{array}{c} 1 + \dfrac{\theta_4(\theta_4 + 1)}{2}V_{020} + \dfrac{\delta_4(\delta_4 + 1)}{2}V_{002} - \theta_4 V_{110} - \delta_4 V_{101} \\ + \theta_4\delta_4 V_{011} \end{array}\right\} - 1\right], \tag{46}$$

$$\mathrm{Bias}\left(\overline{y}_{s_5}^c\right) = \overline{Y}\left[\xi_5\left\{1 + \theta_5^2 V_{020} + \delta_5^2 V_{002} - \theta_5 V_{110} - \delta_5 V_{101} + \theta_5\delta_5 V_{011}\right\} - 1\right]. \tag{47}$$

Proof. The precis of the derivations are given in Appendix *C* for quick review. □

Theorem 2. *The MSE of the proposed combined classes of estimators $\overline{y}_{s_i}^c, i = 1, 2, \ldots, 5$ is given by*

$$MSE\left(\overline{y}_{s_1}^c\right) = \overline{Y}^2\left[\begin{array}{c} 1 + \xi_1^2\left\{1 + V_{200} + 2\theta_1^2 V_{020} + 2\delta_1^2 V_{002} - 2\theta_1 V_{020} - 2\delta_1 V_{002} + 4\theta_1 V_{110} + 4\delta_1 V_{101} + 4\theta_1\delta_1 V_{011}\right\} \\ -2\xi_1\left\{1 + \dfrac{\theta_1^2}{2}V_{020} + \dfrac{\delta_1^2}{2}V_{002} - \theta_1 V_{020} - \delta_1 V_{002} + \theta_1 V_{110} + \delta_1 V_{101} + \theta_1\delta_1 V_{011}\right\} \end{array}\right], \tag{48}$$

$$MSE\left(\overline{y}_{s_2}^c\right) = \overline{Y}^2\left[\begin{array}{c} 1 + \xi_2^2\left\{1 + V_{200} + \theta_2^2 V_{020} + \delta_2^2 V_{002} - \theta_2 V_{020} - \delta_2 V_{002} + 4\theta_2 V_{110} + 4\delta_2 V_{101} + 4\theta_2\delta_2 V_{011}\right\} \\ -2\xi_2\left\{1 - \dfrac{\theta_2}{2}V_{020} - \dfrac{\delta_2}{2}V_{002} + \theta_2 V_{110} + \delta_2 V_{101} + \theta_2\delta_2 V_{011}\right\} \end{array}\right], \tag{49}$$

$$MSE\left(\overline{y}_{s_3}^c\right) = \overline{Y}^2\left[\begin{array}{l} (\xi_3 - 1)^2\overline{Y}^2 + \xi_3^2\overline{Y}^2 V_{200} + \theta_3^2\overline{X}^2 V_{020} + \delta_3^2\overline{Z}^2 V_{002} \\ +2\xi_3\theta_3\overline{X}\overline{Y}V_{110} + 2\xi_3\delta_3\overline{Z}\overline{Y}V_{101} + 2\theta_3\delta_3\overline{X}\overline{Z}V_{011} \end{array}\right], \tag{50}$$

$$MSE\left(\overline{y}_{s_4}^c\right) = \overline{Y}^2\left[\begin{array}{c} 1 + \xi_4^2\left\{1 + V_{200} + \theta_4 V_{020} + \delta_4 V_{002} + 2\theta_4^2 V_{020} + 2\delta_4^2 V_{002} - 4\theta_4 V_{110} - 4\delta_4 V_{101} + 4\theta_4\delta_4 V_{011}\right\} \\ -2\xi_4\left\{1 + \dfrac{\theta_4(\theta_4 + 1)}{2}V_{020} + \dfrac{\delta_4(\delta_4 + 1)}{2}V_{002} - \theta_4 V_{110} - \delta_4 V_{101} + \theta_4\delta_4 V_{011}\right\} \end{array}\right], \tag{51}$$

$$MSE\left(\overline{y}_{s_5}^c\right)=\overline{Y}^2\left[\begin{array}{c}1+\xi_5^2\left\{1+V_{200}+3\theta_5^2V_{020}+3\delta_5^2V_{002}-4\theta_5V_{110}-4\delta_5V_{101}+4\theta_5\delta_5V_{011}\right\}\\ -2\xi_5\left\{1+\theta_5^2V_{020}-\theta_5V_{110}+\delta_5^2V_{002}-\delta_5V_{101}+\theta_5\delta_5V_{011}\right\}\end{array}\right]. \quad (52)$$

Proof. The precis of the derivations are given in Appendix C for quick review. □

Corollary 1. *The minimum MSE of the proposed combined classes of estimators* $\overline{y}_{s_i}^c$, *i = 1,2,..., 5 is given by*

$$\min MSE\left(\overline{y}_{s_i}^c\right)=\overline{Y}^2\left(1-\frac{B_i^2}{A_i}\right);\quad i=1,2,4,5, \quad (53)$$

$$\min MSE\left(\overline{y}_{s_3}^c\right)=\overline{Y}^2\left(1-\xi_{3(opt)}\right)=\overline{Y}^2\left(1-\frac{B_3^2}{A_3}\right). \quad (54)$$

Proof. The precis of derivations and the definition of parametric functions A_i and B_i are given in Appendix C for quick review.

We note that Theorem 2 and Corollary 1 are important to derive the efficiency conditions given in Subsection 4.1. □

3.2. Separate Estimators. We propose some improved separate classes of estimators based on bivariate auxiliary information under *SSRS* as

$$\overline{y}_{s_1}^s=\sum_{h=1}^{L}W_h\xi_{1_h}\overline{y}_h\left[1+\log\left(\frac{\overline{x}_h}{\overline{X}_h}\right)\right]^{\theta_{1_h}}\left[1+\log\left(\frac{\overline{z}_h}{\overline{Z}_h}\right)\right]^{\delta_{1_h}}, \quad (55)$$

$$\overline{y}_{s_2}^s=\sum_{h=1}^{L}W_h\xi_{2_h}\overline{y}_h\left[1+\theta_{2_h}\log\left(\frac{\overline{x}_h}{\overline{X}_h}\right)\right]\left[1+\delta_{2_h}\log\left(\frac{\overline{z}_h}{\overline{Z}_h}\right)\right], \quad (56)$$

$$\overline{y}_{s_3}^s=\sum_{h=1}^{L}W_h\left[\xi_{3_h}\overline{y}_h+\theta_{3_h}\left(\overline{x}_h-\overline{X}_h\right)+\delta_{3_h}\left(\overline{z}_h-\overline{Z}_h\right)\right], \quad (57)$$

$$\overline{y}_{s_4}^s=\sum_{h=1}^{L}W_h\xi_{4_h}\overline{y}_h\left(\frac{\overline{X}_h}{\overline{x}_h}\right)^{\theta_{4_h}}\left(\frac{\overline{Z}_h}{\overline{z}_h}\right)^{\delta_{4_h}}, \quad (58)$$

$$\overline{y}_{s_5}^s=\sum_{h=1}^{L}W_h\xi_{5_h}\overline{y}_h\left[\frac{\overline{X}_h}{\overline{X}_h+\theta_{5_h}\left(\overline{x}_h-\overline{X}_h\right)}\right]\left[\frac{\overline{Z}_h}{\overline{Z}_h+\delta_{5_h}\left(\overline{z}_h-\overline{Z}_h\right)}\right], \quad (59)$$

where ξ_{i_h}, θ_{i_h} and δ_{i_h}, $i=1,2,\ldots,5$ are duly opted scalars in stratum *h*.

Theorem 3. *The bias of the proposed separate classes of estimators* $\overline{y}_{s_i}^s, i=1,2,\ldots,5$ *is given by*

$$\text{Bias}\left(\overline{y}_{s_1}^s\right)=\sum_{h=1}^{L}W_h\overline{Y}_h\left[\xi_{1_h}\left\{\begin{array}{c}1+\left(\frac{\theta_{1_h}^2}{2}-\theta_{1_h}\right)U_{020}+\left(\frac{\delta_{1_h}^2}{2}-\delta_{1_h}\right)U_{002}\\ +\theta_{1_h}U_{110}+\delta_{1_h}U_{101}+\theta_{1_h}\delta_{1_h}U_{011}\end{array}\right\}-1\right], \quad (60)$$

$$\text{Bias}\left(\overline{y}_{s_2}^s\right)=\sum_{h=1}^{L}W_h\overline{Y}_h\left[\xi_{2_h}\left\{1-\frac{\theta_{2_h}}{2}U_{020}-\frac{\delta_{2_h}}{2}U_{002}+\theta_{2_h}U_{110}+\delta_{2_h}U_{101}+\theta_{2_h}\delta_{2_h}U_{011}\right\}-1\right], \quad (61)$$

$$\text{Bias}\left(\overline{y}_{s_3}^s\right)=\sum_{h=1}^{L}W_h\overline{Y}_h\left(\xi_{3_h}-1\right), \quad (62)$$

$$Bias\left(\overline{y}_{s_4}^s\right)=\sum_{h=1}^{L}W_h\overline{Y}_h\left[\xi_{4_h}\left\{\begin{array}{c}1+\frac{\theta_{4_h}\left(\theta_{4_h}+1\right)}{2}U_{020}+\frac{\delta_{4_h}\left(\delta_{4_h}+1\right)}{2}U_{002}-\theta_{4_h}U_{110}\\ -\delta_{4_h}U_{101}+\theta_{4_h}\delta_{4_h}U_{011}\end{array}\right\}-1\right], \quad (63)$$

$$\text{Bias}\left(\overline{y}_{s_5}^s\right)=\sum_{h=1}^{L}W_h\overline{Y}_h\left[\xi_{5_h}\left\{\begin{array}{c}1+\theta_{5_h}^2U_{020}+\delta_{5_h}^2U_{002}-\theta_{5_h}U_{110}-\delta_{5_h}U_{101}\\ +\theta_{5_h}\delta_{5_h}U_{011}\end{array}\right\}-1\right]. \quad (64)$$

Proof. The precis of the derivations are given in Appendix C for quick review □

Theorem 4. *The MSE of the proposed separate classes of estimators $\overline{y}^s_{s_i}, i = 1, 2, \ldots, 5$ is given by*

$$MSE\left(\overline{y}^s_{s_1}\right) = \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left[\begin{array}{c} 1 + \xi^2_{1_h}\left\{\begin{array}{c} 1 + U_{200} + 2\theta^2_{1_h} U_{020} + 2\delta^2_{1_h} U_{002} - 2\delta_{1_h} U_{020} \\ -2\delta_{1_h} U_{002} + 4\theta_{1_h} U_{110} + 4\delta_{1_h} U_{101} + 4\theta_{1_h}\delta_{1_h} U_{011} \end{array}\right\} \\ -2\xi_{1_h}\left\{\begin{array}{c} 1 + \dfrac{\theta^2_{1_h}}{2} U_{020} + \dfrac{\delta^2_{1_h}}{2} U_{002} - \theta_1 U_{020} - \\ \delta_{1_h} U_{002} + \theta_{1_h} U_{110} + \delta_{1_h} U_{101} + \theta_{1_h}\delta_{1_h} U_{011} \end{array}\right\} \end{array}\right], \quad (65)$$

$$MSE\left(\overline{y}^s_{s_2}\right) = \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left[\begin{array}{c} 1 + \xi^2_{2_h}\left\{\begin{array}{c} 1 + U_{200} + \theta^2_{2_h} U_{020} + \delta^2_{2_h} U_{002} - \theta_{2_h} U_{020} \\ -\delta_{2_h} U_{002} + 4\theta_{2_h} U_{110} + 4\delta_{2_h} U_{101} + 4\theta_{2_h}\delta_{2_h} U_{011} \end{array}\right\} \\ -2\xi_{2_h}\left\{1 - \dfrac{\theta_{2_h}}{2} U_{020} - \dfrac{\delta_{2_h}}{2} U_{002} + \theta_{2_h} U_{110} + \delta_{2_h} U_{101} + \theta_{2_h}\delta_{2_h} U_{011}\right\} \end{array}\right], \quad (66)$$

$$MSE\left(\overline{y}^s_{s_3}\right) = \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left[\begin{array}{c} \left(\xi_{3_h} - 1\right)^2 \overline{Y}^2_h + \xi^2_{3_h} \overline{Y}^2_h U_{200} + \theta^2_{3_h} \overline{X}^2_h U_{020} + \delta^2_{3_h} \overline{Z}^2_h U_{002} \\ +2\xi_{3_h}\theta_{3_h} \overline{X}_h \overline{Y}_h U_{110} + 2\xi_{3_h}\delta_{3_h} \overline{Z}_h \overline{Y}_h U_{101} + 2\theta_{3_h}\delta_{3_h} \overline{X}_h \overline{Z}_h U_{011} \end{array}\right], \quad (67)$$

$$MSE\left(\overline{y}^s_{s_4}\right) = \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left[\begin{array}{c} 1 + \xi^2_{4_h}\left\{\begin{array}{c} 1 + U_{200} + \theta_{4_h} U_{020} + \delta_{4_h} U_{002} + 2\theta^2_{4_h} U_{020} \\ +2\delta^2_4 U_{002} - 4\theta_{4_h} U_{110} - 4\delta_{4_h} U_{101} + 4\theta_{4_h}\delta_{4_h} U_{011} \end{array}\right\} \\ -2\xi_{4_h}\left\{\begin{array}{c} 1 + \dfrac{\theta_{4_h}\left(\theta_{4_h} + 1\right)}{2} U_{020} + \dfrac{\delta_{4_h}\left(\delta_{4_h} + 1\right)}{2} U_{002} - \theta_{4_h} U_{110} \\ -\delta_{4_h} U_{101} + \theta_{4_h}\delta_{4_h} U_{011} \end{array}\right\} \end{array}\right], \quad (68)$$

$$MSE\left(\overline{y}^s_{s_5}\right) = \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left[\begin{array}{c} 1 + \xi^2_{5_h}\left\{\begin{array}{c} 1 + U_{200} + 3\theta^2_{5_h} U_{020} + 3\delta^2_{5_h} U_{002} - 4\theta_{5_h} U_{110} \\ -4\delta_{5_h} U_{101} + 4\theta_{5_h}\delta_{5_h} U_{011} \end{array}\right\} \\ -2\xi_{5_h}\left\{\begin{array}{c} 1 + \theta^2_{5_h} U_{020} - \theta_{5_h} U_{110} + \delta^2_{5_h} U_{002} - \delta_{5_h} U_{101} \\ +\theta_{5_h}\delta_{5_h} U_{011} \end{array}\right\} \end{array}\right]. \quad (69)$$

Proof. The precis of the derivations are given in Appendix C for quick review. □

Corollary 2. *The minimum MSE of the proposed separate classes of estimators $\overline{y}^s_{s_i}$;i = 1,2, . . ., 5 is given by*

$$\min MSE\left(\overline{y}^s_{s_i}\right) = \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(1 - \frac{B^2_{i_h}}{A_{i_h}}\right); \quad i = 1, 2, 4, 5, \quad (70)$$

$$\min MSE\left(\overline{y}^s_{s_3}\right) = \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(1 - \xi_{3_h\,(\mathrm{opt})}\right) = \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(1 - \frac{B^2_{3_h}}{A_{3_h}}\right). \quad (71)$$

Proof. The precis of the derivations and definition of parametric functions A_{i_h} and B_{i_h} are given in Appendix □

Proof. C for quick review.

We again note that Theorem 4 and Corollary 2 are important in order to derive the efficiency conditions given in Subsection 4.2. □

4. Efficiency Conditions

In this section, we derive the efficiency conditions under which the proposed combined and separate classes of estimators dominate the existing combined and separate estimators.

4.1. Combined Estimators. On comparing the minimum *MSE* of the proposed combined estimators $\overline{y}_{s_i}^c, i = 1, 2, \ldots, 5$ from (53) and (54) with the minimum *MSE* of existing combined estimators from (A.1), (A.3), (A.4), (A.6), (A.8), (A.10), (A.12), (A.14), (A.16), (A.20), (A.22), (A.24), (A.26), and (A.28), we get the following conditions:

$$MSE\left(\overline{y}_m^c\right) > MSE\left(\overline{y}_{s_i}^c\right) \Rightarrow \frac{B_i^2}{A_i} > 1 - V_{200}, \quad (72)$$

$$MSE\left(\overline{y}_r^c\right) > MSE\left(t_{s_i}^c\right) \Rightarrow \frac{B_i^2}{A_i} > 1 - V_{200} - V_{020} - V_{002} + 2V_{110} + 2V_{101} - 2V_{011}, \quad (73)$$

$$MSE\left(\overline{y}_l^c\right) > MSE\left(\overline{y}_{s_i}^c\right) \Rightarrow \frac{B_i^2}{A_i} > 1 - V_{200} + \frac{\left(V_{002}V_{110}^2 + V_{020}V_{101}^2 - 2V_{110}V_{101}V_{011}\right)}{\left(V_{020}V_{002} - V_{011}^2\right)}, \quad (74)$$

$$MSE\left(\overline{y}_o^c\right) > MSE\left(\overline{y}_{s_i}^c\right) \Rightarrow \frac{B_i^2}{A_i} > 1 - V_{200} - V_{002} - 2V_{101} + \frac{\left(V_{101} - V_{002} - V_{110} + V_{011}\right)^2}{\left(V_{020} + V_{002} - 2V_{011}\right)}, \quad (75)$$

$$MSE\left(\overline{y}_a^c\right) > MSE\left(\overline{y}_{s_i}^c\right) \Rightarrow \frac{B_i^2}{A_i} > 1 - V_{200} + \frac{\left(V_{002}V_{110}^2 + V_{020}V_{101}^2 - 2V_{110}V_{101}V_{011}\right)}{\left(V_{020}V_{002} - V_{011}^2\right)}, \quad (76)$$

$$MSE\left(\overline{y}_w^c\right) > MSE\left(\overline{y}_{s_i}^c\right) \Rightarrow \frac{B_i^2}{A_i} > \left[1 - V_{200} - a_2^2 V_{002} + 2V_{002}V_{101} + \frac{\left(V_{002}V_{110}^2 + V_{020}V_{101}^2 - 2V_{110}V_{101}V_{011}\right)}{\left(V_{020}V_{002} - V_{011}^2\right)}\right], \quad (77)$$

$$MSE\left(\overline{y}_{kc_i}^c\right) > MSE\left(\overline{y}_{s_i}^c\right) \Rightarrow \frac{B_i^2}{A_i} > \left[\begin{array}{c} 1 - V_{200} - \lambda_i^2 V_{020} - \Delta_i^2 V_{002} + 2\lambda_i V_{110} \\ +2\Delta_i V_{101} - 2\lambda_i \Delta_i V_{011} \end{array}\right], \quad (78)$$

$$MSE\left(\overline{y}_s^c\right) > MSE\left(\overline{y}_{s_i}^c\right) \Rightarrow \frac{B_i^2}{A_i} > \frac{\left(A_2 - A_3 A_4\right)}{A_1 A_2 - A_3^2} - \frac{A_4\left(A_1 A_4 - A_3\right)}{A_1 A_2 - A_3^2}, \quad (79)$$

$$MSE\left(\overline{y}_{kk}^c\right) > MSE\left(\overline{y}_{s_i}^c\right) \Rightarrow \frac{B_i^2}{A_i} > 1 - V_{200} + \frac{\left(V_{002}V_{110}^2 + V_{020}V_{101}^2 - 2V_{110}V_{101}V_{011}\right)}{\left(V_{020}V_{002} - V_{011}^2\right)}, \quad (80)$$

$$MSE\left(\overline{y}_{tc}^c\right) > MSE\left(\overline{y}_{s_i}^c\right) \Rightarrow \frac{B_i^2}{A_i} > \left[\begin{array}{c} 1 - k^{*^2}\left\{V_{200} + \frac{1}{4}\left(V_{020} + V_{002} - 2V_{002}\right) + V_{101} - V_{110}\right\} \\ -2k^*\left(k^* - 1\right)\left\{\frac{1}{8}\left(3V_{020} - V_{002} - 2V_{011}\right)\right\} - \left(k^* - 1\right)^2 \end{array}\right], \quad (81)$$

$$MSE\left(\overline{y}_{l_1}^c\right) > MSE\left(\overline{y}_{s_i}^c\right) \Rightarrow \frac{B_i^2}{A_i} > 1 - V_{200} + \frac{\left(V_{101}^2 V_{020} + V_{110}^2 V_{002} - 2V_{011}V_{101}V_{110}\right)}{V_{020}V_{002}\left(1 - \rho_{xz_h}^2\right)}, \quad (82)$$

$$MSE\left(\overline{y}_{l_2}^c\right) > MSE\left(\overline{y}_{s_i}^c\right) \Rightarrow \frac{B_i^2}{A_i} > 1 - V_{200} + \frac{\left(V_{002}V_{110}^2 + V_{020}V_{101}^2 - 2V_{110}V_{101}V_{011}\right)}{\left(V_{020}V_{002} - V_{011}^2\right)}, \quad (83)$$

$$MSE\left(\overline{y}_{mu}^{c}\right) > MSE\left(\overline{y}_{s_i}^{c}\right) \Rightarrow \frac{B_i^2}{A_i} > \frac{V_{011}^2}{4V_{020}} + \frac{A_m^2}{B_m}, \tag{84}$$

$$MSE\left(\overline{y}_{mu_1}^{c}\right) > MSE\left(\overline{y}_{s_i}^{c}\right) \Rightarrow \frac{B_i^2}{A_i} > 2O - P, \tag{85}$$

$$MSE\left(\overline{y}_{mu_2}^{c}\right) > MSE\left(\overline{y}_{s_i}^{c}\right) \Rightarrow \frac{B_i^2}{A_i} > \frac{O^2}{P}. \tag{86}$$

If conditions (72) to (86) hold, then the proposed combined classes of estimators $\overline{y}_{s_i}^{c}, i = 1, 2, \ldots, 5$ perform better than the other existing combined estimators.

4.2. Separate Estimators. On comparing the minimum *MSE* of the proposed separate estimators $\overline{y}_{s_i}^{s}, i = 1, 2, \ldots, 5$ from (69) and (70) with the minimum *MSE* of the existing separate estimators from (B.1), (B.3), (B.4), (B.6), (B.8), (B.10), (B.12), (B.14), (B.16), (B.18), (B.20), (B.22), (B.24), and (B.26), we get the following efficiency conditions:

$$\begin{gathered} MSE\left(\overline{y}_{m}^{s}\right) > MSE\left(\overline{y}_{s_i}^{s}\right), \\ \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(1 - \frac{B_{i_h}^2}{A_{i_h}}\right) < \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 U_{200}, \end{gathered} \tag{87}$$

$$\begin{gathered} MSE\left(\overline{y}_{r}^{s}\right) > MSE\left(\overline{y}_{s_i}^{s}\right), \\ \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(1 - \frac{B_{i_h}^2}{A_{i_h}}\right) < \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(U_{200} + U_{020} + U_{002} - 2U_{110} - 2U_{101} + 2U_{011}\right), \end{gathered} \tag{88}$$

$$\begin{gathered} MSE\left(\overline{y}_{l}^{s}\right) > MSE\left(\overline{y}_{s_i}^{s}\right), \\ \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(1 - \frac{B_{i_h}^2}{A_{i_h}}\right) < \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left\{U_{200} - \frac{\left(U_{002}U_{110}^2 + U_{020}U_{101}^2 - 2U_{110}U_{101}U_{011}\right)}{\left(U_{020}U_{002} - U_{011}^2\right)}\right\}, \end{gathered} \tag{89}$$

$$\begin{gathered} MSE\left(\overline{y}_{o}^{s}\right) > MSE\left(\overline{y}_{s_i}^{s}\right), \\ \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(1 - \frac{B_{i_h}^2}{A_{i_h}}\right) < \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left\{U_{200} + U_{002} + 2U_{101} - \frac{\left(U_{101} - U_{002} - U_{110} + U_{011}\right)^2}{\left(U_{020} + U_{002} - 2U_{011}\right)}\right\}, \end{gathered} \tag{90}$$

$$\begin{gathered} MSE\left(\overline{y}_{a}^{s}\right) > MSE\left(\overline{y}_{s_i}^{s}\right), \\ \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(1 - \frac{B_{i_h}^2}{A_{i_h}}\right) < \sum_{h=1}^{L} W_h^2 \overline{Y}^2 \left\{U_{200} - \frac{\left(U_{002}U_{110}^2 + U_{020}U_{101}^2 - 2U_{110}U_{101}U_{011}\right)}{\left(U_{020}U_{002} - U_{011}^2\right)}\right\}, \end{gathered} \tag{91}$$

$$\begin{gathered} MSE\left(\overline{y}_{w}^{s}\right) > MSE\left(\overline{y}_{s_i}^{s}\right), \\ \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(1 - \frac{B_{i_h}^2}{A_{i_h}}\right) < \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left\{\left(U_{200} + a_{2_h}^2 U_{002} - 2U_{002}U_{101}\right) - \frac{\left(U_{002}U_{110}^2 + U_{020}U_{101}^2 - 2U_{110}U_{101}U_{011}\right)}{\left(U_{020}U_{002} - U_{011}^2\right)}\right\}, \end{gathered} \tag{92}$$

$$\begin{gathered} MSE\left(\overline{y}_{kc_i}^{s}\right) > MSE\left(\overline{y}_{s_i}^{s}\right), \\ \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(1 - \frac{B_{i_h}^2}{A_{i_h}}\right) < \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(\begin{matrix} U_{200} + \lambda_{i_h}^2 U_{020} + \Delta_{i_h}^2 U_{002} - 2\lambda_{i_h} U_{110} \\ -2\Delta_{i_h} U_{101} + 2\lambda_{i_h}\Delta_{i_h} U_{011} \end{matrix}\right), \end{gathered} \tag{93}$$

$$MSE\left(\overline{y}_s^s\right) > MSE\left(\overline{y}_{s_i}^s\right),$$
$$\sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(1 - \frac{B_{i_h}^2}{A_{i_h}}\right) < \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left\{1 - \frac{\left(A_{2_h} - A_{3_h} A_{4_h}\right)}{\left(A_{1_h} A_{2_h} - A_{3_h}^2\right)} - \frac{A_{4_h}\left(A_{1_h} A_{4_h} - A_{3_h}\right)}{\left(A_{1_h} A_{2_h} - A_{3_h}^2\right)}\right\}, \tag{94}$$

$$MSE\left(\overline{y}_{kk}^s\right) > MSE\left(\overline{y}_{s_i}^s\right),$$
$$\sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(1 - \frac{B_{i_h}^2}{A_{i_h}}\right) < \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left\{U_{200} - \frac{\left(U_{002} U_{110}^2 + U_{020} U_{101}^2 - 2U_{110} U_{101} U_{011}\right)}{\left(U_{020} U_{002} - U_{011}^2\right)}\right\}, \tag{95}$$

$$MSE\left(\overline{y}_{tc}^s\right) > MSE\left(\overline{y}_{s_i}^s\right),$$
$$\sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(1 - \frac{B_{i_h}^2}{A_{i_h}}\right) < \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left[\begin{array}{c} k_h^{*^2} \left\{\begin{array}{c} U_{200} + \frac{1}{4}\left(U_{020} + U_{002} - 2U_{002}\right) \\ +U_{101} - U_{110} \end{array}\right\} \\ +2k_h^*\left(k_h^* - 1\right)\left\{\frac{1}{8}\left(3U_{020} - U_{002} - 2U_{011}\right) + \frac{1}{2}\left(U_{101} - U_{110}\right)\right\} + \left(k_h^* - 1\right)^2 \end{array}\right], \tag{96}$$

$$MSE\left(\overline{y}_{l_1}^s\right) > MSE\left(\overline{y}_{s_i}^s\right),$$
$$\sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(1 - \frac{B_{i_h}^2}{A_{i_h}}\right) < \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left\{U_{200} - \frac{\left(U_{101}^2 U_{020} + U_{110}^2 U_{002} - 2U_{011} U_{101} U_{110}\right)}{U_{020} U_{002}\left(1 - \rho_{xz_h}^2\right)}\right\}, \tag{97}$$

$$MSE\left(\overline{y}_{l_2}^s\right) > MSE\left(\overline{y}_{s_i}^s\right),$$
$$\sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(1 - \frac{B_{i_h}^2}{A_{i_h}}\right) < \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left\{U_{200} - \frac{\left(U_{002} U_{110}^2 + U_{020} U_{101}^2 - 2U_{110} U_{101} U_{011}\right)}{\left(U_{020} U_{002} - U_{011}^2\right)}\right\}, \tag{98}$$

$$MSE\left(\overline{y}_{mu}^s\right) > MSE\left(\overline{y}_{s_i}^s\right),$$
$$\sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(1 - \frac{B_{i_h}^2}{A_{i_h}}\right) < \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(1 - \frac{U_{011}^2}{4U_{020}} - \frac{A_{m_h}^2}{B_{m_h}}\right), \tag{99}$$

$$MSE\left(\overline{y}_{mu_1}^s\right) > MSE\left(\overline{y}_{s_i}^s\right),$$
$$\sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(1 - \frac{B_{i_h}^2}{A_{i_h}}\right) < \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(1 + P_h - 2O_h\right), \tag{100}$$

$$MSE\left(\overline{y}_{mu_2}^s\right) > MSE\left(\overline{y}_{s_i}^s\right),$$
$$\sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(1 - \frac{B_{i_h}^2}{A_{i_h}}\right) < \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(1 - \frac{O_h^2}{P_h}\right). \tag{101}$$

If the conditions (87) to (101) hold then the proposed separate classes of estimators $\overline{y}_{s_i}^s, i = 1, 2, \ldots, 5$ perform better than the other existing separate estimators.

4.3. Comparison of Proposed Combined and Separate Estimators. By comparing the minimum *MSE* of the proposed combined and separate classes of estimators $\overline{y}_{s_i}^c$ and $\overline{y}_{s_i}^s$, $i = 1, 2, \ldots, 5$, we get

Table 1: *ARB* and *PRE* of different combined estimators.

ρ_{xy}	0.6		0.7		0.8		0.9		0.9		0.8		0.7		0.6	
ρ_{yz}	0.6		0.7		0.8		0.9		0.8		0.7		0.6		0.5	
ρ_{xz}	0.6		0.7		0.8		0.9		0.7		0.6		0.5		0.4	
Estimators	*ARB*	*PRE*	*ARB*	*PRE*	*ARB*	*PRE*	*ARB*	*PRE*	*ARB*	*PRE*	*ARB*	*PRE*	*ARB*	*PRE*	*ARB*	*PRE*
$\overline{y}_m^c$	0	100	0	100	0	100	0	100	0	100	0	100	0	100	0	100
$\overline{y}_r^c$	1	96.00	1	117.56	1	154.49	1	211.78	1	236.59	1	168.39	1	127.97	1	102.94
$\overline{y}_l^c$	0	178.81	0	230.11	0	329.15	0	564.57	0	550.00	0	321.64	0	230.37	0	175.72
$\overline{y}_o^c$	3.87	178.77	3.46	230.43	3.14	328.86	2.92	562.99	2.95	541.45	3.19	320.36	3.53	229.21	4.02	175.53
$\overline{y}_a^c$	18.22	178.81	25.21	230.11	44.60	329.15	207.86	564.57	103.66	550.00	33.90	321.64	20.58	230.37	15.25	175.72
$\overline{y}_w^c$	18.98	130.63	13.23	139.16	10.43	143.77	9.14	119.34	6.40	406.71	9.89	230.53	14.63	172.55	24.54	143.63
$\overline{y}_{kc_1}^c$	2.04	97.44	2.33	119.64	2.67	157.84	3.08	218.06	3.32	243.37	2.86	171.81	2.45	130.03	2.11	104.34
$\overline{y}_{kc_2}^c$	2.77	116.00	3.29	147.34	3.99	202.00	4.88	307.31	5.25	338.91	4.24	217.45	3.46	155.73	2.86	122.12
$\overline{y}_{kc_3}^c$	9.67	164.20	16.15	219.66	45.50	327.67	81.99	646.17	72.23	594.80	48.80	314.14	17.00	213.07	9.63	161.47
$\overline{y}_{kc_4}^c$	2.01	96.45	2.29	118.20	2.62	155.57	3.01	213.84	3.25	238.74	2.81	169.44	2.41	128.61	2.08	103.36
$\overline{y}_s^c$	3.29	156.26	4.33	196.09	6.39	277.78	12.81	526.31	12.29	526.31	6.10	277.78	4.09	196.09	3.15	156.26
$\overline{y}_{kk}^c$	9.36	178.81	7.67	230.11	6.47	329.15	5.71	564.57	5.39	550.00	6.14	321.64	7.28	230.37	9.17	175.72
$\overline{y}_{tc}^c$	1.74	85.04	1.89	86.93	2.03	88.40	2.23	89.99	2.16	90.63	1.96	89.92	1.80	88.57	1.69	87.28
$\overline{y}_{l_1}^c$	9.90	178.81	8.00	230.11	6.73	329.15	5.87	564.57	4.07	550.00	4.79	321.64	5.76	230.37	7.18	175.72
$\overline{y}_{l_2}^c$	6.76	178.81	9.24	230.11	14.17	329.15	28.50	564.57	14.63	550.00	9.47	321.64	6.91	230.37	5.35	175.72
$\overline{y}_{mu}^c$	3.78	180.61	5.19	236.10	7.96	345.39	16.52	674.09	16.29	731.60	7.47	348.51	4.74	230.25	3.47	176.36
$\overline{y}_{mu_1}^c$	3.56	180.89	5.99	236.77	8.21	346.11	17.11	674.87	17.09	732.41	7.76	349.12	5.16	231.58	4.37	176.72
$\overline{y}_{mu_2}^c$	3.77	180.96	5.87	236.97	8.01	346.78	17.27	675.17	17.17	733.01	7.94	349.24	5.24	232.19	4.20	176.87
$\overline{y}_{s_1}^c$	21.07	182.84	14.45	239.56	5.45	349.67	3.45	678.58	3.18	750.01	5.09	351.68	14.23	234.40	16.98	178.75
$\overline{y}_{s_2}^c$	1.27	182.78	1.38	238.11	1.98	348.07	2.61	678.00	1.22	748.16	1.08	351.24	1.06	234.23	12	178.67
$\overline{y}_{s_3}^c$	3.83	181.98	5.22	238.54	7.97	347.88	16.52	678.17	17.52	749.21	7.73	351.45	4.89	234.38	3.61	178.73
$\overline{y}_{s_4}^c$	3.83	182.83	5.22	239.27	7.97	349.36	16.52	678.44	17.52	749.91	7.73	351.61	4.89	234.39	3.61	178.74
$\overline{y}_{s_5}^c$	3.82	181.92	5.21	238.54	7.97	347.88	16.52	678.17	17.52	749.20	7.73	351.45	4.89	234.23	3.60	178.73

$$\min MSE\left(\overline{y}_{s_i}^c\right) - \min MSE\left(\overline{y}_{s_i}^s\right) = \sum_{h=1}^{L}\left[\left(\overline{Y}^2 - W_h^2\overline{Y}_h^2\right) - \left(\overline{Y}^2\frac{B_i^2}{A_i} - W_h^2\overline{Y}_h^2\frac{B_{i_h}^2}{A_{i_h}}\right)\right]. \tag{102}$$

If the ratio estimate is veritable and the relationship between auxiliary and study variables within each stratum is a straight line passing through origin then the last term of (102) is broadly small and it vanished.

Furthermore, unless R_h is invariant from stratum to stratum, separate estimators probably become more efficient in each stratum if the sample in each stratum is large enough so that the approximate formula for $MSE\,(\overline{y}_{s_i}^s), i = 1, 2, \ldots, 5$ is valid and the cumulative bias that can affect the proposed estimators is negligible, whereas the proposed combined estimators are to be preferably recommended with only a small sample in each stratum ([29]).Furthermore, the conditions of Subsection 4.1, Subsection 4.2, and Subsection 4.3 are held in practice by being verified through a simulation study.

5. Simulation Study

To enhance the credibility of the theoretical development of the proposed combined and separate classes of estimators, we have conducted a simulation study. In the procedure, the following steps are considered:

(i) Generate trivariate random observations of size $N = 2000$ units using a trivariate normal distribution in *R* software with parameters $\overline{Y} = 20$, $\overline{X} = 25$, $\overline{Z} = 30$, $\sigma_y = 5$, $\sigma_x = 6$, $\sigma_z = 7$ and different amounts of correlation coefficients ρ_{xy}, ρ_{yz}, and ρ_{xz}.

(ii) Stratify the above population into 4 equal disjoint strata and quantify a sample of size $n = 50$ units from each stratum.

(iii) Tabulate all necessary statistics.

(iv) Using 15,000 iterations to calculate the absolute relative bias (*ARB*) and percent relative efficiency (*PRE*) of various combined and separate classes of estimators. The *ARB* and *PRE* of different estimators *T* are calculated regarding the classical ratio and usual mean estimators and results are reported in Table 1 and Table 2. The *ARB* and *PRE* are calculated using the following expressions.

$$ARB = \left|\frac{\sum_{i=1}^{15,000}\left(\overline{y}_r - \overline{Y}\right)}{\sum_{i=1}^{15,000}\left(T - \overline{Y}\right)}\right|, \tag{103}$$

$$PRE = \frac{\sum_{i=1}^{15,000}\left(\overline{y}_m - \overline{Y}\right)^2}{\sum_{i=1}^{15,000}\left(T - \overline{Y}\right)^2} \times 100, \tag{104}$$

Table 2: *ARB* and *PRE* of different separate estimators.

ρ_{xy}	0.6		0.7		0.8		0.9		0.9		0.8		0.7		0.6	
ρ_{yz}	0.6		0.7		0.8		0.9		0.8		0.7		0.6		0.5	
ρ_{xz}	0.6		0.7		0.8		0.9		0.7		0.6		0.5		0.4	
Estimators	*ARB*	*PRE*	*ARB*	*PRE*	*ARB*	*PRE*	*ARB*	*PRE*	*ARB*	*PRE*	*ARB*	*PRE*	*ARB*	*PRE*	*ARB*	*PRE*
$\overline{y}^s_m$	0	100	0	100	0	100	0	100	0	100	0	100	0	100	0	100
$\overline{y}^s_r$	1	92.12	1	119.54	1	154.96	1	212.28	1	236.66	1	167.43	1	128.43	1	103.45
$\overline{y}^s_l$	0	179.22	0	230.97	0	329.79	0	565.16	0	550.80	0	322.11	0	230.06	0	179.23
$\overline{y}^s_o$	3.86	179.56	3.45	230.95	3.15	328.24	2.95	563.35	2.97	542.16	3.20	320.77	3.51	230.00	4.00	177.21
$\overline{y}^s_a$	18.19	179.22	25.24	230.97	44.63	329.79	207.92	565.16	103.86	550.80	34.33	322.11	21.76	230.06	15.23	179.23
$\overline{y}^s_w$	19.00	1310	13.29	139.89	10.88	144.86	9.15	120.85	6.44	408.54	9.93	2311	14.77	171.73	24.64	142.28
$\overline{y}^s_{kc_1}$	2.10	98.00	2.31	120.20	2.77	158.88	3.12	218.99	3.37	244.06	2.89	172.44	2.49	130.88	2.14	103.48
$\overline{y}^s_{kc_2}$	2.75	116.92	3.40	148.73	4.00	202.91	4.90	308.00	5.31	342.15	4.30	219.35	3.36	151.53	2.70	120.77
$\overline{y}^s_{kc_3}$	9.60	167.45	16.19	221.34	46.00	327.89	82.08	648.01	72.45	596.60	48.89	313.31	17.11	210.98	9.43	160.64
$\overline{y}^s_{kc_4}$	2.12	96.87	2.30	117.99	2.59	155.02	3.09	214.47	3.30	242.00	2.88	173.07	2.45	127.76	2.09	104.55
$\overline{y}^s_s$	3.31	156.86	4.35	196.97	6.47	279.00	12.94	527.21	12.37	528.21	6.12	277.87	4.15	196.83	3.19	156.99
$\overline{y}^s_{kk}$	9.35	179.22	7.66	230.97	6.46	329.79	5.73	565.16	5.42	550.80	6.15	322.11	7.31	230.06	9.19	179.23
$\overline{y}^s_{tc}$	1.76	85.91	1.91	86.99	2.09	88.88	2.26	90.43	2.10	90.98	1.99	89.98	1.84	89.25	1.73	88.03
$\overline{y}^s_{l_1}$	9.93	179.22	8.11	230.97	6.77	329.79	5.89	565.16	4.13	550.80	4.82	322.11	5.70	230.06	7.20	179.23
$\overline{y}^s_{l_2}$	6.80	179.22	9.29	230.97	15.17	329.79	29.00	565.16	15.66	550.80	9.44	322.11	6.96	230.06	5.39	178.23
$\overline{y}^s_{mu}$	3.84	181.09	5.23	236.97	7.87	346.65	16.71	674.85	16.38	732.28	7.51	348.81	4.83	232.03	3.52	178.24
$\overline{y}^s_{mu_1}$	3.62	181.24	6.09	237.84	8.24	347.15	17.13	675.51	17.13	732.76	7.80	349.95	5.20	232.13	4.50	177.89
$\overline{y}^s_{mu_2}$	3.84	181.85	5.90	237.57	8.20	349.17	17.76	676.74	17.10	732.99	7.98	351.53	5.32	234.11	4.23	176.97
$\overline{y}^s_{s_1}$	20.76	183.64	14.47	2415	5.44	352.99	3.46	679.74	3.20	751.32	5.15	353.82	14.67	236.78	17.00	182.52
$\overline{y}^s_{s_2}$	1.28	182.85	1.39	240.07	1.30	350.99	6.61	678.68	1.25	750.73	10	352.46	11	235.52	14	180.76
$\overline{y}^s_{s_3}$	3.88	182.17	5.25	239.47	8.07	350.59	17.00	678.96	17.59	749.83	7.80	352.69	4.95	235.75	3.63	180.77
$\overline{y}^s_{s_4}$	3.88	182.91	5.25	239.98	8.07	351.87	16.90	679.03	17.60	750.79	7.81	352.96	4.91	236.04	3.60	1817
$\overline{y}^s_{s_5}$	3.86	182.16	5.22	239.47	8.06	350.59	17.01	678.96	17.59	749.82	7.880	352.69	4.95	235.75	3.62	180.77

where $T = \overline{y}^c_m, \overline{y}^c_r, \overline{y}^c_l, \overline{y}^c_o, \overline{y}^c_m, \overline{y}^c_a, \overline{y}^c_w, \overline{y}^c_{kc_i}, i = 1,2,3,4, \overline{y}^c_s, \overline{y}^c_{tc}, \overline{y}^c_{mu}, \overline{y}^c_{mu_1}, \overline{y}^c_{mu_2}, \overline{y}^c_{s_i}, i = 1,2,\ldots,5, \overline{y}^s_m, \overline{y}^s_r, \overline{y}^s_l, \overline{y}^s_o, \overline{y}^s_m, \overline{y}^s_a, \overline{y}^s_w, \overline{y}^s_{kc_i}, i = 1,2,3,4, \overline{y}^s_s, \overline{y}^s_{tc}, \overline{y}^s_{mu}, \overline{y}^s_{mu_1}, \overline{y}^s_{mu_2}$ and $\overline{y}^s_{s_i}, i = 1,2,\ldots,5$.

The simulation findings of the combined and separate classes of estimators are exposited in terms of *ARB* and *PRE* in Table 1 and Table 2 for different values of correlation coefficients. The results exhibit the dominance of the proposed combined and separate classes of estimators $\overline{y}^c_{s_i}$ and $\overline{y}^s_{s_i}$, $i = 1,2,\ldots,5$, respectively, over the combined and separate usual mean estimators, classical ratio and regression estimators, Olkin [24] type estimator, Abu–Dayyeh et al. [25] type estimators, Kadilar and Cingi [7] type estimator, Singh et al. [16] estimator, Koyuncu and Kadilar [12] estimator, Tailor and Chouhan [14] estimator, Lone et al. [18, 19] estimators and Muneer et al. [21, 22] estimators in terms of *PRE*. Also, the proposed combined and separate class of estimators $\overline{y}^c_{s_1}$ and $\overline{y}^s_{s_1}$ are found to be most efficient among the proposed combined and separate classes of estimators for each passably chosen values of correlation coefficients.

6. Conclusion

In this article, we propose some improved classes of estimators for population mean by extending the work of Bhushan et al. [28] using bivariate auxiliary information under *SSRS*. The mathematical expressions of bias and *MSE* of the proposed classes of estimators are obtained up to the first order of approximation. The efficiency conditions are derived under which the suggested estimators perform better than the other existing estimators. In support of the theoretical results, a simulation study is carried out using an artificially generated population with various amounts of correlation coefficients. From the perusal of the theoretical and simulation results reported in Table 1 and Table 2, we conclude that:

(i) The proposed combined classes of estimators $\overline{y}^c_{s_i}, i = 1,2,\ldots,5$ perform better than the combined form of usual mean estimator $\overline{y}^c_m$, classical ratio and regression estimators $\overline{y}^c_r$ & $\overline{y}^c_l$, Olkin [24] type estimator $\overline{y}^c_o$, Abu–Dayyeh et al. [25] type estimators $\overline{y}^c_a$ & $\overline{y}^c_w$, Kadilar and Cingi [7] type estimators $\overline{y}^c_{kc_i}, i = 1,2,3,4$, Koyuncu and Kadilar [12] estimator $\overline{y}^c_{kk}$, Singh et al. [16] estimator $\overline{y}^c_s$, Tailor and Chouhan [14] estimator $\overline{y}^c_{tc}$, Lone et al. [18, 19] estimators $\overline{y}^c_{l_i}, i = 1,2$ and Muneer et al. [21, 22] estimators $\overline{y}^c_{mu}$ & $\overline{y}^c_{mu_i}, i = 1,2$ for different values of correlation coefficients.

(ii) The proposed separate classes of estimators $\overline{y}^s_{s_i}, i = 1,2,\ldots,5$ dominate the separate form of usual mean estimator $\overline{y}^s_m$, classical ratio and regression estimators $\overline{y}^s_r$ & $\overline{y}^s_l$, Olkin [24] type estimator $\overline{y}^s_o$, Abu-Dayyeh et al. [25] type estimators $\overline{y}^s_a$ & $\overline{y}^s_w$, Kadilar and Cingi [7] type estimators $\overline{y}^s_{kc_i}, i = 1,2,3,4$, Koyuncu and Kadilar [12] estimator $\overline{y}^s_{kk}$, Singh et al. [16] estimator $\overline{y}^s_s$, Tailor and Chouhan [14] estimator $\overline{y}^s_{tc}$, Lone et al. [18, 19] estimators $\overline{y}^s_{l_i}, i = 1,2$ and Muneer et al. [21, 22]

estimators $\overline{y}_{mu}^{s}$ & $\overline{y}_{mu_i}^{s}$, $i = 1, 2$ for different values of correlation coefficients.

(iii) Since, the proposed combined and separate classes of estimators $\overline{y}_{s_i}^{c}$ and $\overline{y}_{s_i}^{s}$, $i = 1, 2, \ldots, 5$ are, respectively, superior to the combined and separate ratio and chain ratio exponential estimators $\overline{y}_{mu_1}^{c}$, $\overline{y}_{mu_1}^{s}$ and $\overline{y}_{mu_2}^{c}$, $\overline{y}_{mu_2}^{s}$ envisaged by Muneer et al. [22] consequently the proposed combined and separate classes of estimators $\overline{y}_{s_i}^{c}$ and $\overline{y}_{s_i}^{s}$, $i = 1, 2, \ldots, 5$ will also dominate those 65 estimators that are the members of the combined and separate ratio and chain ratio exponential estimators $\overline{y}_{mu_1}^{c}$, $\overline{y}_{mu_2}^{c}$ and $\overline{y}_{mu_1}^{s}$, $\overline{y}_{mu_2}^{s}$, respectively.

(iv) The proposed combined and separate class of estimators $\overline{y}_{s_1}^{c}$ and $\overline{y}_{s_1}^{s}$ perform better among the proposed classes of estimators.

(v) The proposed separate classes of estimators $\overline{y}_{s_i}^{s}$, $i = 1, 2, \ldots, 5$ dominate the proposed combined classes of estimators $\overline{y}_{s_i}^{c}$, $i = 1, 2, \ldots, 5$ in terms of greater *PRE* for various amounts of correlation coefficients.

(vi) The *PRE* of the proposed combined and separate classes of estimators increases as the values of correlation coefficients increase and vice versa.

Thus, the proposed combined and separate classes of estimators can be preferably used by the survey professionals in practice.

Appendix

A. Bias and MSE of the Existing Combined Estimators

The bias and *MSE* expressions of the existing combined estimators are expressed as mentioned below:

$$MSE\left(\overline{y}_{m}^{c}\right) = \overline{Y}^{2} V_{200}, \tag{A.1}$$

$$\text{Bias}\left(\overline{y}_{r}^{c}\right) = \overline{Y}\left[V_{020} + V_{002} + V_{011} - V_{101} + V_{110}\right], \tag{A.2}$$

$$MSE\left(\overline{y}_{r}^{c}\right) = \overline{Y}^{2}\left[+V_{020} + V_{002} - 2V_{110} - 2V_{101} + 2V_{011}\right], \tag{A.3}$$

$$\min MSE\left(\overline{y}_{l}^{c}\right) = \overline{Y}^{2}\left[V_{200} - \frac{\left(V_{002}V_{110}^{2} + V_{020}V_{101}^{2} - 2V_{110}V_{101}V_{011}\right)}{\left(V_{020}V_{002} - V_{011}^{2}\right)}\right], \tag{A.4}$$

$$\text{Bias}\left(\overline{y}_{0}^{c}\right) = \overline{Y}\left[w\left(V_{110} - V_{101}\right) + V_{101}\right], \tag{A.5}$$

$$\min MSE\left(\overline{y}_{0}^{c}\right) = \overline{Y}^{2}\left[\left(V_{200} + V_{002} + 2V_{101}\right) - \frac{\left(V_{101} - V_{002} - V_{110} + V_{011}\right)^{2}}{\left(V_{020} + V_{002} - 2V_{011}\right)}\right], \tag{A.6}$$

$$Bias\left(\overline{y}_{a}^{c}\right) = \overline{Y}\left[\frac{a_1\left(a_1 + 1\right)}{2}V_{020} + \frac{a_2\left(a_2 + 1\right)}{2}V_{002} + a_1 a_2 V_{011} - a_1 V_{110} - a_2 V_{101}\right], \tag{A.7}$$

$$\min MSE\left(\overline{y}_{a}^{c}\right) = \overline{Y}^{2}\left[V_{200} - \frac{\left(V_{002}V_{110}^{2} + V_{020}V_{101}^{2} - 2V_{110}V_{101}V_{011}\right)}{\left(V_{020}V_{002} - V_{011}^{2}\right)}\right], \tag{A.8}$$

$$\text{Bias}\left(\overline{y}_{w}^{c}\right) = \overline{Y}\left[w_1 a_1 V_{110} + w_2 a_2 V_{101} + w_1\frac{a_1\left(a_1 - 1\right)}{2}V_{020} + w_2\frac{a_2\left(a_2 - 1\right)}{2}V_{002}\right], \tag{A.9}$$

$$\min MSE\left(\overline{y}_{w}^{c}\right) = \overline{Y}^{2}\left[V_{200} + a_2^{2}V_{002} - 2V_{002}V_{101} - \frac{\left(V_{002}V_{110}^{2} + V_{020}V_{101}^{2} - 2V_{110}V_{101}V_{011}\right)}{\left(V_{020}V_{002} - V_{011}^{2}\right)}\right], \tag{A.10}$$

$$\text{Bias}\left(\overline{y}_{kc_i}^{c}\right) = \overline{Y}\left[\lambda_i^{2}V_{020} + \Delta_i^{2}V_{002} - \lambda_i V_{110} - \Delta_i V_{101} + \lambda_i \Delta_i V_{011}\right]; \quad i = 1, 2, 3, 4, \tag{A.11}$$

$$MSE\left(\overline{y}_{kc_i}^{c}\right) = \overline{Y}^{2}\left[V_{200} + \lambda_i^{2}V_{020} + \Delta_i^{2}V_{002} - 2\lambda_i V_{110} - 2\Delta_i V_{101} + 2\lambda_i \Delta_i V_{011}\right], \tag{A.12}$$

$$\text{Bias}\left(\overline{y}_s^c\right) = \overline{Y}\left[\eta_1 + \eta_2\left\{\begin{array}{c} 1 + \dfrac{\alpha(\alpha+1)}{2}V_{020} + \dfrac{\beta(\beta-1)}{2}V_{002} \\ -\alpha V_{110} + \beta V_{101} - \alpha\beta V_{011}\end{array}\right\} - 1\right], \tag{A.13}$$

$$\min MSE\left(\overline{y}_s^c\right) = \overline{Y}^2\left[1 - \frac{(A_2 - A_3A_4)}{A_1A_2 - A_3^2} - \frac{A_4(A_1A_4 - A_3)}{A_1A_2 - A_3^2}\right], \tag{A.14}$$

$$\text{Bias}\left(\overline{y}_{kk}^c\right) = \overline{Y}\left[\begin{array}{c} -g_1v_1\lambda_1V_{110} - g_2v_2\lambda_2V_{101} + \lambda_1\lambda_2g_1g_2v_1v_2V_{011} \\ +\dfrac{g_1(g_1+1)}{2}\lambda_1^2v_1^2V_{020} + \dfrac{g_2(g_2+1)}{2}\lambda_2^2v_2^2V_{002}\end{array}\right], \tag{A.15}$$

$$\min MSE\left(\overline{y}_{kk}^c\right) = \overline{Y}^2\left[V_{200} - \frac{\left(V_{002}V_{110}^2 + V_{020}V_{101}^2 - 2V_{110}V_{101}V_{011}\right)}{\left(V_{020}V_{002} - V_{011}^2\right)}\right], \tag{A.16}$$

$$\text{Bias}\left(\overline{y}_{tc}^c\right) = \overline{Y}(k^* - 1) + k^*\overline{Y}\left[\frac{1}{8}(3V_{020} - V_{002} - 2V_{011}) + \frac{1}{2}(V_{101} - V_{110})\right], \tag{A.17}$$

$$\min MSE\left(\overline{y}_{tc}^c\right) = \overline{Y}^2\left[\begin{array}{c} k^{*^2}\left\{V_{200} + \dfrac{1}{4}(V_{020} + V_{002} - 2V_{002}) + V_{101} - V_{110}\right\} + (k^* - 1)^2 \\ \left\{\dfrac{1}{8}(3V_{020} - V_{002} - 2V_{011}) + \dfrac{1}{2}(V_{101} - V_{110})\right\}\end{array}\right], \tag{A.18}$$

$$\text{Bias}\left(\overline{y}_{l_1}^c\right) = \overline{Y}\left[\begin{array}{c} \dfrac{L_1}{4}\left(\dfrac{L_1}{2} - 1\right)V_{020} + \dfrac{L_2}{4}\left(\dfrac{L_2}{2} - 1\right)V_{002} - \dfrac{L_1L_2}{4}V_{011} \\ +\dfrac{1}{2}(L_2V_{101} - L_1V_{110})\end{array}\right], \tag{A.19}$$

$$\min MSE\left(\overline{y}_{l_1}^c\right) = \overline{Y}^2\left[V_{200} - \frac{\left(V_{101}^2V_{020} + V_{110}^2V_{002} - 2V_{011}V_{101}V_{110}\right)}{\left(V_{020}V_{002} - V_{011}^2\right)}\right], \tag{A.20}$$

$$\text{Bias}\left(\overline{y}_{l_2}^c\right) = \overline{Y}[V_{020} - V_{011} + V_{101} - V_{110}], \tag{A.21}$$

$$\min MSE\left(\overline{y}_{l_2}^c\right) = \overline{Y}^2\left[V_{200} - \frac{\left(V_{002}V_{110}^2 + V_{020}V_{101}^2 - 2V_{110}V_{101}V_{011}\right)}{\left(V_{020}V_{002} - V_{011}^2\right)}\right], \tag{A.22}$$

$$\text{Bias}\left(\overline{y}_{mu}^c\right) = (k_3 - 1)\overline{Y} + k_3\overline{Y}\left\{\left(\frac{3}{8} - \frac{\alpha}{4}\right)V_{002} - \frac{V_{101}}{2}\right\} + \frac{k_4}{2}\overline{X}V_{011}, \tag{A.23}$$

$$\min MSE\left(\overline{y}_{mu}^c\right) = \overline{Y}^2\left[1 - \frac{V_{011}^2}{4V_{020}} - \frac{A_m^2}{B_m}\right], \tag{A.24}$$

$$\text{Bias}\left(\overline{y}_{mu_1}^c\right) = \overline{Y}\left[\begin{array}{c} -\left(\alpha_1g_1 + \dfrac{\alpha_3}{2}\right)V_{110} - \left(\alpha_2g_2 + \dfrac{\alpha_4}{2}\right)V_{101} \\ +\left\{\dfrac{\alpha_1(\alpha_1+1)}{2}g_1^2 + \dfrac{3\alpha_3(\alpha_3+1)}{8} + \dfrac{\alpha_1\alpha_3g_1}{2}\right\}V_{020} \\ +\left\{\dfrac{\alpha_2(\alpha_2+1)}{2}g_2^2 + \dfrac{3\alpha_4(\alpha_4+1)}{8} + \dfrac{\alpha_2\alpha_4g_2}{2}\right\}V_{002} \\ +\left(\alpha_1\alpha_2g_1g_2 + \dfrac{1}{2}\alpha_2\alpha_3g_2 + \dfrac{1}{2}\alpha_1\alpha_4g_1 + \dfrac{\alpha_3\alpha_4}{4}\right)V_{011}\end{array}\right], \tag{A.25}$$

$$MSE\left(\overline{y}^{c}_{mu_1}\right)=\overline{Y}^2\left[1+P-2O\right], \tag{A.26}$$

$$\text{Bias}\left(\overline{y}^{c}_{mu_2}\right)=(k_1-1)\overline{Y}+k_1\overline{Y}\begin{bmatrix} -\left(\alpha_1 g_1+\dfrac{\alpha_3}{2}\right)V_{110}-\left(\alpha_2 g_2+\dfrac{\alpha_4}{2}\right)V_{101}+ \\ \left\{\dfrac{\alpha_1(\alpha_1+1)}{2}g_1^2+\dfrac{3\alpha_3(\alpha_3+1)}{8}+\dfrac{\alpha_1\alpha_3 g_1}{2}\right\}V_{020}+ \\ \left\{\dfrac{\alpha_2(\alpha_2+1)}{2}g_2^2+\dfrac{3\alpha_4(\alpha_4+1)}{8}+\dfrac{\alpha_2\alpha_4 g_2}{2}\right\}V_{002}+ \\ \left(\alpha_1\alpha_2 g_1 g_2+\dfrac{1}{2}\alpha_2\alpha_3 g_2+\dfrac{1}{2}\alpha_1\alpha_4 g_1+\dfrac{\alpha_3\alpha_4}{4}\right)V_{011} \end{bmatrix}, \tag{A.27}$$

$$\min MSE\left(\overline{y}^{c}_{mu_2}\right)=\overline{Y}^2\left[1-\frac{O^2}{P}\right], \tag{A.28}$$

where$\lambda_1=\sum_{h=1}^{L}W_h\overline{X}_h/\sum_{h=1}^{L}W_h(\overline{X}_h+C_{x_h})$; $\lambda_2=\sum_{h=1}^{L}W_h\overline{X}_h/\sum_{h=1}^{L}(\overline{X}_h+\beta_2(x_h))$; $\lambda_3=\sum_{h=1}^{L}W_h\overline{X}_hC_{x_h}/\sum_{h=1}^{L}W_h(\overline{X}_hC_{x_h}+\beta_2(x_h))$; $\lambda_4=\sum_{h=1}^{L}W_h\overline{X}_h\beta_2(x_h)/\sum_{h=1}^{L}W_h(\overline{X}_h\beta_2(x_h)+C_{x_h})$; $\Delta_1=\sum_{h=1}^{L}W_h\overline{Z}_h/\sum_{h=1}^{L}W_h(\overline{Z}_h+C_{z_h})$; $\Delta_2=\sum_{h=1}^{L}W_h\overline{Z}_h/\sum_{h=1}^{L}W_h(\overline{Z}_h+\beta_2(z_h))$; $\Delta_3=\sum_{h=1}^{L}W_h\overline{Z}_hC_{z_h}/\sum_{h=1}^{L}W_h(\overline{Z}_hC_{z_h}+\beta_2(z_h))$; $\Delta_4=\sum_{h=1}^{L}W_h\overline{Z}_h\beta_2(z_h)/\sum_{h=1}^{L}W_h(\overline{Z}_h\beta_2(z_h)+C_{z_h})$; $A_1=1+V_{200}$; $A_2=1+V_{200}+\alpha(2\alpha+1)V_{020}+\delta(2\delta-1)V_{002}-4\alpha V_{110}+4\delta V_{101}-4\alpha\delta V_{011}$; $A_3=1+V_{200}+\{(\alpha(\alpha+1))/2\}V_{020}+\{(\delta(\delta+1))/2\}V_{002}-2\alpha V_{110}+2\delta V_{101}-\alpha\delta V_{011}$; $A_4=1+\{(\alpha(\alpha+1))/2\}V_{020}+\{(\delta(\delta-1))/2\}V_{002}-\alpha V_{110}+\delta V_{101}-\alpha\delta V_{011}$; $P=1+R+2S$; $O=1+S$; $R=V_{200}+\{\alpha_1 g_1+(\alpha_3/2)\}^2V_{020}+\{\alpha_2 g_2+(\alpha_4/2)\}^2V_{002}-2\{\alpha_1 g_1+(\alpha_3/2)\}V_{110}-2(\alpha_2 g_2+(\alpha_4/2))V_{101}+2\{\alpha_1 g_1+(\alpha_3/2/)\}\{\alpha_2 g_2+(\alpha_4/2)\}V_{011}$ and $S=-\{\alpha_1 g_1+(\alpha_3/2)\}V_{110}-\{\alpha_2 g_2+(\alpha_4/2)\}V_{101}+[\alpha_1\{(\alpha_1+1)g_1^2/2\}+3\{\alpha_3(\alpha_3+1)/8\}+(\alpha_1\alpha_3 g_1/2)]U_{020}+[\{\alpha_2(\alpha_2+1)g_2^2/2\}+\{3\alpha_4(\alpha_4+1)/8\}+\{\alpha_2\alpha_4 g_2/2\}]V_{002}+\{\alpha_1\alpha_2 g_1 g_2+(\alpha_2\alpha_3 g_2/2)\}V_{011}+\{(\alpha_1\alpha_4 g_1/2)+(\alpha_3\alpha_4/4)\}V_{011}$.

The optimum values of the scalars are given as

$$\beta_{(\text{opt})}=\frac{\overline{Y}}{\overline{X}}\left[\frac{V_{002}V_{110}-V_{101}V_{011}}{V_{020}V_{002}-V_{011}^2}\right], \tag{A.29}$$

$$\theta_{(\text{opt})}=\frac{\overline{Y}}{\overline{Z}}\left[\frac{V_{020}V_{101}-V_{110}V_{011}}{V_{020}V_{002}-V_{011}^2}\right], \tag{A.30}$$

$$w_{(\text{opt})}=-\left[\frac{V_{101}-V_{002}-V_{110}+V_{011}}{V_{020}+V_{002}-2V_{011}}\right], \tag{A.31}$$

$$a_{1(\text{opt})}=\left[\frac{V_{002}V_{110}-V_{101}V_{011}}{V_{020}V_{002}-V_{011}^2}\right], \tag{A.32}$$

$$a_{2(\text{opt})}=\left[\frac{V_{020}V_{101}-V_{110}V_{011}}{V_{020}V_{002}-V_{011}^2}\right], \tag{A.33}$$

$$w_{1(\text{opt})}=\left[\frac{a_2^2V_{002}+a_1^2V_{110}-a_2V_{101}-a_1a_2V_{011}}{a_1^2V_{020}+a_2^2V_{002}-2a_1a_2V_{011}}\right], \tag{A.34}$$

$$w_{2(\text{opt})}=1-w_{1(\text{opt})}, \tag{A.35}$$

$$\eta_{1(\text{opt})}=\frac{A_2-A_3A_4}{A_1A_2-A_3^2}, \tag{A.36}$$

$$\eta_{2(\text{opt})} = \frac{A_1A_4 - A_3}{A_1A_2 - A_3^2}, \tag{A.37}$$

$$l_{1(\text{opt})} = \left[\frac{V_{002}V_{110} - V_{101}V_{011}}{lV_{020}V_{002} - V_{011}^2}\right], \tag{A.38}$$

$$l_{2(\text{opt})} = \left[\frac{V_{020}V_{101} - V_{110}V_{011}}{lV_{020}V_{002} - V_{011}^2}\right], \tag{A.39}$$

$$k_{(\text{opt})} = \frac{\left(1 + \{(1/8)(3V_{020} - V_{002} - 2V_{011}) + (1/2)(V_{101} - V_{110})\}\right)}{\begin{pmatrix} 1 + 2\{(1/8)(3V_{020} - V_{002} - 2V_{011}) + (1/2)(V_{101} - V_{110})\} \\ +k^2\{V_{200} + (1/4)(V_{020} + V_{002} - 2V_{002}) + V_{101} - V_{110}\} \end{pmatrix}} = k^* \text{ (say)}, \tag{A.40}$$

$$L_{1(\text{opt})} = \frac{2(V_{002}V_{110} - V_{101}V_{011})}{(V_{020}V_{002} - V_{011}^2)}, \tag{A.41}$$

$$L_{2(\text{opt})} = \frac{2(V_{110}V_{011} - V_{020}V_{101})}{(V_{020}V_{002} - V_{011}^2)}, \tag{A.42}$$

$$\lambda_{(\text{opt})} = \frac{(V_{002}V_{110} - V_{101}V_{011})}{(V_{020}V_{002} - V_{011}^2)}, \tag{A.43}$$

$$\psi_{(\text{opt})} = \frac{-(V_{020}V_{101} - V_{110}V_{011})}{(V_{020}V_{002} - V_{011}^2)}, \tag{A.44}$$

$$k_{3(\text{opt})} = \frac{1 + (3/8 - \alpha/4)V_{002} - (V_{101}/2) - (V_{011}(V_{011} - V_{110})/2V_{020})}{1 + V_{200} + (1 - \alpha/2)V_{002} - 2V_{101} - \left((V_{011} - V_{110})^2/V_{020}\right)} = \frac{A_m}{B_m} \text{ (say)}, \tag{A.45}$$

$$k_{4(\text{opt})} = \frac{\overline{Y}}{\overline{X}}\left[\frac{V_{011}}{2V_{020}} - k_3\left(\frac{V_{011} - V_{110}}{V_{020}}\right)\right], \tag{A.46}$$

$$k_{1(\text{opt})} = \frac{O}{P}. \tag{A.47}$$

B. Bias and MSE of the Existing Separate Estimators

The bias and *MSE* expressions of the existing separate estimators are expressed as

$$MSE(\overline{y}_m^s) = \sum_{h=1}^{L} W_h^2\overline{Y}_h^2 U_{200}, \tag{B.1}$$

$$\text{Bias}(\overline{y}_r^s) = \sum_{h=1}^{L} W_h\overline{Y}_h[U_{020} + U_{002} + U_{011} - U_{101} + U_{110}], \tag{B.2}$$

$$MSE(\overline{y}_r^s) = \sum_{h=1}^{L} W_h^2\overline{Y}_h^2[U_{200} + U_{020} + U_{002} - 2U_{110} - 2U_{101} + 2U_{011}], \tag{B.3}$$

$$\min MSE\left(\overline{y}_l^s\right) = \sum_{h=1}^{L} W_h^2\overline{Y}_h^2\left[U_{200} - \frac{\left(U_{002}U_{110}^2 + U_{020}U_{101}^2 - 2U_{110}U_{101}U_{011}\right)}{\left(U_{020}U_{002} - U_{011}^2\right)}\right], \quad \text{(B.4)}$$

$$\text{Bias}\left(\overline{y}_0^s\right) = \sum_{h=1}^{L} W_h\overline{Y}_h\left[w_h\left(U_{110} - U_{101}\right) + U_{101}\right], \quad \text{(B.5)}$$

$$\min MSE\left(\overline{y}_o^s\right) = \sum_{h=1}^{L} W_h^2\overline{Y}_h^2\left[\left(U_{200} + U_{002} + 2U_{101}\right) - \frac{\left(U_{101} - U_{002} - U_{110} + U_{011}\right)^2}{\left(U_{020} + U_{002} - 2U_{011}\right)}\right], \quad \text{(B.6)}$$

$$Bias\left(\overline{y}_a^s\right) = \sum_{h=1}^{L} W_h\overline{Y}_h\left[\begin{array}{c}\frac{a_{1_h}\left(a_{1_h}+1\right)}{2}U_{020} + \frac{a_{2_h}\left(a_{2_h}+1\right)}{2}U_{002} + a_{1_h}a_{2_h}U_{011}\\ -a_{1_h}U_{110} - a_{2_h}U_{101}\end{array}\right], \quad \text{(B.7)}$$

$$\min MSE\left(\overline{y}_a^s\right) = \sum_{h=1}^{L} W_h^2\overline{Y}^2\left[U_{200} - \frac{\left(U_{002}U_{110}^2 + U_{020}U_{101}^2 - 2U_{110}U_{101}U_{011}\right)}{\left(U_{020}U_{002} - U_{011}^2\right)}\right], \quad \text{(B.8)}$$

$$\text{Bias}\left(\overline{y}_w^s\right) = \sum_{h=1}^{L} W_h\overline{Y}_h\left[\begin{array}{c}+w_{2_h}a_{2_h}U_{101} + w_{1_h}\frac{a_{1_h}\left(a_{1_h}-1\right)}{2}U_{020}\\ +w_{2_h}\frac{a_{2_h}\left(a_{2_h}-1\right)}{2}U_{002}\end{array}\right], \quad \text{(B.9)}$$

$$\min MSE\left(\overline{y}_w^s\right) = \sum_{h=1}^{L} W_h^2\overline{Y}_h^2\left[\begin{array}{c}U_{200} + a_{2_h}^2U_{002} - 2U_{002}U_{101}\\ -\frac{\left(U_{002}U_{110}^2 + U_{020}U_{101}^2 - 2U_{110}U_{101}U_{011}\right)}{\left(U_{020}U_{002} - U_{011}^2\right)}\end{array}\right], \quad \text{(B.10)}$$

$$\text{Bias}\left(\overline{y}_{kc_i}^s\right) = \sum_{h=1}^{L} W_h\overline{Y}_h\left[\lambda_{i_h}^2U_{020} + \Delta_{i_h}^2U_{002} - \lambda_{i_h}U_{110} - \Delta_{i_h}U_{101} + \lambda_{i_h}\Delta_{i_h}U_{011}\right]; \quad i = 1, 2, 3, 4, \quad \text{(B.11)}$$

$$MSE\left(\overline{y}_{kc_i}^s\right) = \sum_{h=1}^{L} W_h^2\overline{Y}_h^2\left[U_{200} + \lambda_{i_h}^2U_{020} + \Delta_{i_h}^2U_{002} - 2\lambda_{i_h}U_{110} - 2\Delta_{i_h}U_{101} + 2\lambda_{i_h}\Delta_{i_h}U_{011}\right], \quad \text{(B.12)}$$

$$\text{Bias}\left(\overline{y}_s^s\right) = \sum_{h=1}^{L} W_h\overline{Y}_h\left[\eta_{1_h} + \eta_{2_h}\left\{\begin{array}{c}1 + \frac{\alpha_h\left(\alpha_h+1\right)}{2}U_{020} + \frac{\beta_h\left(\beta_h-1\right)}{2}U_{002}\\ -\alpha_hU_{110} + \beta_hU_{101} - \alpha_h\beta_hU_{011}\end{array}\right\} - 1\right], \quad \text{(B.13)}$$

$$\min MSE\left(\overline{y}_s^s\right) = \sum_{h=1}^{L} W_h^2\overline{Y}_h^2\left[1 - \frac{\left(A_{2_h} - A_{3_h}A_{4_h}\right)}{A_{1_h}A_{2_h} - A_{3_h}^2} - \frac{A_{4_h}\left(A_{1_h}A_{4_h} - A_{3_h}\right)}{A_{1_h}A_{2_h} - A_{3_h}^2}\right], \quad \text{(B.14)}$$

$$\text{Bias}\left(\overline{y}_{kk}^s\right) = \sum_{h=1}^{L} W_h\overline{Y}_h\left[\begin{array}{c}-g_{1_h}v_{1_h}\lambda_{1_h}U_{110} - g_{2_h}v_{2_h}\lambda_{2_h}U_{101} + \lambda_{1_h}\lambda_{2_h}g_{1_h}g_{2_h}v_{1_h}v_{2_h}U_{011}\\ +\frac{g_{1_h}\left(g_{1_h}+1\right)}{2}\lambda_{1_h}^2v_{1_h}^2U_{020} + \frac{g_{2_h}\left(g_{2_h}+1\right)}{2}\lambda_{2_h}^2v_{2_h}^2U_{002}\end{array}\right], \quad \text{(B.15)}$$

$$\min MSE\left(\overline{y}_{kk}^s\right) = \sum_{h=1}^{L} W_h^2\overline{Y}_h^2\left[U_{200} - \frac{\left(U_{002}U_{110}^2 + U_{020}U_{101}^2 - 2U_{110}U_{101}U_{011}\right)}{\left(U_{020}U_{002} - U_{011}^2\right)}\right], \quad \text{(B.16)}$$

$$\text{Bias}\left(\bar{y}_{tc}^{s}\right)=\sum_{h=1}^{L}W_h\bar{Y}_h\left(k_h^{*}-1\right)+k_h^{*}\bar{Y}_h\left[\frac{1}{8}\left(3U_{020}-U_{002}-2U_{011}\right)+\frac{1}{2}\left(U_{101}-U_{110}\right)\right], \tag{B.17}$$

$$\min MSE\left(\bar{y}_{tc}^{s}\right)=\sum_{h=1}^{L}W_h^2\bar{Y}_h^2\left[\begin{array}{c}k_h^{*^2}\left\{U_{200}+\frac{1}{4}\left(U_{020}+U_{002}-2U_{002}\right)+U_{101}-U_{110}\right\}\\ +2k_h^{*}\left(k_h^{*}-1\right)\left\{\frac{1}{8}\left(3U_{020}-U_{002}-2U_{011}\right)+\frac{1}{2}\left(U_{101}-U_{110}\right)\right\}\\ +\left(k_h^{*}-1\right)^2\end{array}\right], \tag{B.18}$$

$$\text{Bias}\left(\bar{y}_{l_1}^{s}\right)=\sum_{h=1}^{L}W_h\bar{Y}_h\left[\begin{array}{c}\frac{L_{1_h}}{4}\left(\frac{L_{1_h}}{2}-1\right)U_{020}+\frac{L_{2_h}}{4}\left(\frac{L_{2_h}}{2}-1\right)U_{002}-\frac{L_{1_h}L_{2_h}}{4}U_{011}\\ +\frac{1}{2}\left(L_{2_h}U_{101}-L_{1_h}U_{110}\right)\end{array}\right], \tag{B.19}$$

$$\min MSE\left(\bar{y}_{l_1}^{s}\right)=\sum_{h=1}^{L}W_h^2\bar{Y}_h^2\left[U_{200}-\frac{\left(U_{101}^2U_{020}+U_{110}^2U_{002}-2U_{011}U_{101}U_{110}\right)}{\left(U_{020}U_{002}-U_{011}^2\right)}\right], \tag{B.20}$$

$$\text{Bias}\left(\bar{y}_{l_2}^{s}\right)=\sum_{h=1}^{L}W_h\bar{Y}_h\left[U_{020}-U_{011}+U_{101}-U_{110}\right], \tag{B.21}$$

$$\min MSE\left(\bar{y}_{l_2}^{s}\right)=\sum_{h=1}^{L}W_h^2\bar{Y}_h^2\left[U_{200}-\frac{\left(U_{002}U_{110}^2+U_{020}U_{101}^2-2U_{110}U_{101}U_{011}\right)}{\left(U_{020}U_{002}-U_{011}^2\right)}\right], \tag{B.22}$$

$$Bias\left(\bar{y}_{mu}^{s}\right)=\sum_{h=1}^{L}W_h\bar{Y}_h\left[\left(k_{3_h}-1\right)+k_{3_h}\left\{\left(\frac{3}{8}-\frac{\alpha_h}{4}\right)U_{002}-\frac{U_{101}}{2}\right\}+\frac{k_{4_h}}{2}\frac{\bar{X}_h}{\bar{Y}_h}U_{011}\right], \tag{B.23}$$

$$\min MSE\left(\bar{y}_{mu}^{s}\right)=\sum_{h=1}^{L}W_h^2\bar{Y}_h^2\left[1-\frac{U_{011}^2}{4U_{020}}-\frac{A_{m_h}^2}{B_{m_h}}\right], \tag{B.24}$$

$$\text{Bias}\left(\bar{y}_{mu_1}^{s}\right)=\sum_{h=1}^{L}W_h\bar{Y}_h\left[\begin{array}{c}-\left(\alpha_{1_h}g_{1_h}+\frac{\alpha_{3_h}}{2}\right)U_{110}-\left(\alpha_{2_h}g_{2_h}+\frac{\alpha_{4_h}}{2}\right)U_{101}+\\ \left\{\frac{\alpha_{1_h}\left(\alpha_{1_h}+1\right)}{2}g_{1_h}^2+\frac{3\alpha_{3_h}\left(\alpha_{3_h}+1\right)}{8}+\frac{\alpha_{1_h}\alpha_{3_h}g_{1_h}}{2}\right\}U_{020}+\\ \left\{\frac{\alpha_{2_h}\left(\alpha_{2_h}+1\right)}{2}g_{2_h}^2+\frac{3\alpha_{4_h}\left(\alpha_{4_h}+1\right)}{8}+\frac{\alpha_{2_h}\alpha_{4_h}g_{2_h}}{2}\right\}U_{002}+\\ \left(\alpha_{1_h}\alpha_{2_h}g_{1_h}g_{2_h}+\frac{1}{2}\alpha_{2_h}\alpha_{3_h}g_{2_h}+\frac{1}{2}\alpha_{1_h}\alpha_{4_h}g_{1_h}+\frac{\alpha_{3_h}\alpha_{4_h}}{4}\right)U_{011}\end{array}\right], \tag{B.25}$$

$$MSE\left(\bar{y}_{mu_1}^{s}\right)=\sum_{h=1}^{L}W_h^2\bar{Y}_h^2\left[1+P_h-2O_h\right], \tag{B.26}$$

$$\mathrm{Bias}\left(\overline{y}^{s}_{mu_2}\right)=\left[\sum_{h=1}^{L}W_h\left(k_{1_h}-1\right)\overline{Y}_h+\sum_{h=1}^{L}W_hk_{1_h}\overline{Y}_h\left[\begin{array}{c}-\left(\alpha_{1_h}g_{1_h}+\dfrac{\alpha_{3_h}}{2}\right)U_{110}-\left(\alpha_{2_h}g_{2_h}+\dfrac{\alpha_{4_h}}{2}\right)U_{101}+\\ \left\{\dfrac{\alpha_{1_h}\left(\alpha_{1_h}+1\right)}{2}g_{1_h}^2+\dfrac{3\alpha_{3_h}\left(\alpha_{3_h}+1\right)}{8}+\dfrac{\alpha_{1_h}\alpha_{3_h}g_{1_h}}{2}\right\}U_{020}+\\ \left\{\dfrac{\alpha_{2_h}\left(\alpha_{2_h}+1\right)}{2}g_{2_h}^2+\dfrac{3\alpha_{4_h}\left(\alpha_{4_h}+1\right)}{8}+\dfrac{\alpha_{2_h}\alpha_{4_h}g_{2_h}}{2}\right\}U_{002}+\\ \left(\begin{array}{c}\alpha_{1_h}\alpha_{2_h}g_{1_h}g_{2_h}+\dfrac{1}{2}\alpha_{2_h}\alpha_{3_h}g_{2_h}\\ +\dfrac{1}{2}\alpha_{1_h}\alpha_{4_h}g_{1_h}+\dfrac{\alpha_{3_h}\alpha_{4_h}}{4}\end{array}\right)U_{011}\end{array}\right]\right], \tag{B.27}$$

$$\min MSE\left(\overline{y}^{s}_{mu_2}\right)=\sum_{h=1}^{L}W_h^2\overline{Y}_h^2\left[1-\frac{O_h^2}{P_h}\right], \tag{B.28}$$

where $\lambda_{1_h}=\overline{X}_h/(\overline{X}_h+C_{x_h})$; $\lambda_{2_h}=\overline{X}_h/(\overline{X}_h+\beta_2(x_h))$; $\lambda_{3_h}=\overline{X}_hC_{x_h}/(\overline{X}_hC_{x_h}+\beta_2(x_h))$; $\lambda_{4_h}=\overline{X}_h\beta_2(x_h)/(\overline{X}_h\beta_2(x_h)+C_{x_h})$; $\Delta_{1_h}=\overline{Z}_h/(\overline{Z}_h+C_{z_h})$; $\Delta_{2_h}=\overline{Z}_h/(\overline{Z}_h+\beta_2(z_h))$; $\Delta_{3_h}=\overline{Z}_hC_{z_h}/(\overline{Z}_hC_{z_h}+\beta_2(z_h))$; $\Delta_{4_h}=\overline{Z}_h\beta_2(z_h)/(\overline{Z}_h\beta_2(z_h)+C_{z_h})$; $A_{1_h}=1+U_{200}$; $A_{2_h}=1+U_{200}+\alpha_h(2\alpha+1)U_{020}+\delta_h(2\delta_h-1)U_{002}-4\alpha_hU_{110}+4\delta_hU_{101}-4\alpha_h\delta_hU_{011}$; $A_{3_h}=1+U_{200}+\{\{(\alpha_h(\alpha_h+1))/2\}U_{020}+\{(\delta_h(\delta_h+1))/2\}\}U_{002}-2\alpha_hU_{110}+2\delta_hU_{101}-\alpha_h\delta_hU_{011}$; $A_{4_h}=1+\{\alpha_h(\alpha_h+1)/2\}U_{020}+\{\delta_h(\delta_h-1)/2\}U_{002}-\alpha_hU_{110}+\delta_hU_{101}-\alpha_h\delta_hU_{011}$; $P_h=1+R_h+2S_h$; $O_h=1+S_h$; $R_h=U_{200}+\{\alpha_{1_h}g_{1_h}+(\alpha_{3_h}/2)\}^2U_{020}+\{\alpha_{2_h}g_{2_h}+(\alpha_{4_h}/2)\}^2U_{002}-2\{\alpha_{1_h}g_{1_h}+(\alpha_{3_h}/2)\}U_{110}-2\{\alpha_{2_h}g_{2_h}+(\alpha_{4_h}/2)\}U_{101}+2\{\alpha_{1_h}g_{1_h}+(\alpha_{3_h}/2)\}(\alpha_{2_h}g_{2_h}+(1/2)\alpha_{4_h})U_{011}$; $S_h=-(\alpha_{1_h}g_{1_h}+(\alpha_{3_h}/2))U_{110}+\{\alpha_{1_h}((\alpha_{1_h}+1)g_{1_h}^2/2)+3\alpha_{3_h}((\alpha_{3_h}+1)/8)+(\alpha_{1_h}\alpha_{3_h}g_{1_h})/2\}U_{020}+\{\alpha_{2_h}((\alpha_{2_h}+1)g_{2_h}^2)/2+3\alpha_{4_h}((\alpha_{4_h}+1)/8)+(\alpha_{2_h}\alpha_{4_h}g_{2_h}/2)\}U_{002}-(\alpha_{2_h}g_{2_h}+(\alpha_{4_h}/2))U_{101}+\{\alpha_{1_h}\alpha_{2_h}g_{1_h}g_{2_h}+(\alpha_{2_h}\alpha_{3_h}g_{2_h}/2)+(\alpha_{1_h}\alpha_{4_h}g_{1_h}/2)+(\alpha_{3_h}\alpha_{4_h}/4)\}U_{011}$.

The optimum values of scalars are tabulated as

$$\beta_{h(\mathrm{opt})}$$

$$\theta_{h(\mathrm{opt})}=\frac{\overline{Y}_h}{\overline{Z}_h}\left[\frac{U_{020}U_{101}-U_{110}U_{011}}{U_{020}U_{002}-U_{011}^2}\right],$$

$$w_{h(\mathrm{opt})}=-\left[\frac{U_{101}-U_{002}-U_{110}+U_{011}}{U_{020}+U_{002}-2U_{011}}\right],$$

$$a_{1_h(\mathrm{opt})}=\left[\frac{U_{002}U_{110}-U_{101}U_{011}}{U_{020}U_{002}-U_{011}^2}\right],a_{2_h(\mathrm{opt})}=\left[\frac{U_{020}U_{101}-U_{110}U_{011}}{U_{020}U_{002}-U_{011}^2}\right],w_{1_h(\mathrm{opt})}=\left[\frac{a_{2_h}^2U_{002}+a_{1_h}^2U_{110}-a_{2_h}U_{101}-a_{1_h}a_{2_h}U_{011}}{a_{1_h}^2U_{020}+a_{2_h}^2U_{002}-2a_{1_h}a_{2_h}U_{011}}\right],w_{2_h(\mathrm{opt})}=1-w_{1_h(\mathrm{opt})},\eta_{1_h(\mathrm{opt})}=\frac{A_{2_h}-A_{3_h}A_{4_h}}{A_{1_h}A_{2_h}-A_{3_h}^2},\eta_{2_h(\mathrm{opt})}=\frac{A_{1_h}A_{4_h}-A_{3_h}}{A_{1_h}A_{2_h}-A_{3_h}^2},l_{1_h(\mathrm{opt})}=\left[\frac{U_{002}U_{110}-U_{101}U_{011}}{U_{020}U_{002}-U_{011}^2}\right],l_{2_h(\mathrm{opt})}=\left[\frac{U_{020}U_{101}-U_{110}U_{011}}{U_{020}U_{002}-U_{011}^2}\right], \tag{B.29}$$

$$
\begin{aligned}
&k_{h(\text{opt})}\\
&L_{1_h(\text{opt})} = \frac{2\left(U_{002}U_{110} - U_{101}U_{011}\right)}{\left(U_{020}U_{002} - U_{011}^2\right)},\\
&L_{2_h(\text{opt})} = \frac{2\left(U_{110}U_{011} - U_{020}U_{101}\right)}{\left(U_{020}U_{002} - U_{011}^2\right)},\\
&\lambda_{h(\text{opt})} = \frac{\left(U_{002}U_{110} - U_{101}U_{011}\right)}{\left(U_{020}U_{002} - U_{011}^2\right)},\\
&\psi_{h(\text{opt})} = \frac{-\left(U_{020}U_{101} - U_{110}U_{011}\right)}{\left(U_{020}U_{002} - U_{011}^2\right)},\\
&k_{3_h(\text{opt})} = \frac{1 + ((3/8) - (\alpha_h/4))U_{002} - (U_{101}/2) - (U_{011}(U_{011} - U_{110})/2U_{020})}{1 + U_{200} + (1 - (\alpha_h/2))U_{002} - 2U_{101} - \left(\left(U_{011} - U_{110}\right)^2/U_{020}\right)} = \frac{A_{m_h}}{B_{m_h}} \text{ (say)},\\
&k_{4_h(\text{opt})} = \frac{\overline{Y}_h}{\overline{X}_h}\left[\frac{U_{011}}{2U_{020}} - k_{3_h(o)p(t)}\left(\frac{U_{011} - U_{110}}{U_{020}}\right)\right],\\
&k_{1_h(\text{opt})} = \frac{O_h}{P_h}.
\end{aligned}
\tag{B.30}
$$

C. Bias and MSE of the Proposed Combined and Separate Estimators

This section addresses the precis of the proof of Theorem 2 and Corollary 1 of Subsection 3.1.

Consider the estimator

$$\overline{y}_{s_3}^c = \xi_3\overline{y}_{st} + \theta_3\left(\overline{x}_{st} - \overline{X}\right) + \delta_3\left(\overline{z}_{st} - \overline{Z}\right). \tag{C.1}$$

Using the notations defined in the earlier section, we express the estimator $\overline{y}_{s_3}^c$ in terms ε*/s* as

$$\overline{y}_{s_3}^c - \overline{Y} = \overline{Y}\left[\,(\xi_3 - 1)\overline{Y} + \overline{Y}\xi_3\varepsilon_0 + \theta_3\overline{X}\varepsilon_1 + \delta_3\overline{Z}\varepsilon_2\,\right]. \tag{C.2}$$

Now, squaring and taking expectations both sides of (C.2), we will get the *MSE* of the estimator as

$$MSE\left(\overline{y}_{s_3}^c\right) = \overline{Y}^2\left[\begin{array}{l}(\xi_3 - 1)^2\overline{Y}^2 + \xi_3^2\overline{Y}^2V_{200} + \theta_3^2\overline{X}^2V_{020} + \delta_3^2\overline{Z}^2V_{002}\\ +2\xi_3\theta_3\overline{XY}V_{110} + 2\xi_3\delta_3\overline{ZY}V_{101} + 2\theta_3\delta_3\overline{XZ}V_{011}\end{array}\right]. \tag{C.3}$$

The optimum values of ξ_3, θ_3, and δ_3 can be obtained by minimizing (C.3) w.r.t ξ_3, θ_3, and δ_3 as

$$\xi_{3(\text{opt})} = \frac{1}{\left[1 + V_{200} - \left(V_{020}V_{101}^2 + V_{002}V_{110}^2 - 2V_{110}V_{101}V_{011}/\left(V_{020}V_{002} - V_{011}^2\right)\right)\right]} = \frac{B_3}{A_3} \text{ (say)}, \tag{C.4}$$

$$\theta_{3(\text{opt})} = \xi_{3(\text{opt})}\left(\frac{\overline{Y}}{\overline{X}}\right)\frac{\left(V_{101}V_{011} - V_{002}V_{110}\right)}{\left(V_{020}V_{002} - V_{011}^2\right)}, \tag{C.5}$$

$$\delta_{3(\text{opt})} = \xi_{3(\text{opt})}\left(\frac{\overline{Y}}{\overline{Z}}\right)\frac{\left(V_{110}V_{011} - V_{020}V_{101}\right)}{\left(V_{020}V_{002} - V_{011}^2\right)}. \tag{C.6}$$

Putting $\xi_{3(\text{opt})}$, $\theta_{3(\text{opt})}$ and $\delta_{3(\text{opt})}$ in (C.3), we get the minimum *MSE* as

$$\min MSE\left(\overline{y}_{s_3}^c\right) = \overline{Y}^2\left(1 - \xi_{3(\text{opt})}\right) = \overline{Y}^2\left(1 - \frac{B_3^2}{A_3}\right). \tag{C.7}$$

In similar way, we can tabulate the *MSE* of other estimators $\overline{y}_{s_i}^c, i = 1, 2, 4, 5$ as

$$MSE\left(\overline{y}_{s_i}^c\right) = \overline{Y}^2\left[\,1 + \xi_i^2A_i - 2\xi_iB_i\,\right]. \tag{C.8}$$

The minimum MSE of the estimators $\overline{y}_{s_i}^c, i = 1, 2, 4, 5$ is given by

$$\min MSE\left(\overline{y}_{s_i}^c\right) = \overline{Y}^2\left(1 - \frac{B_i^2}{A_i}\right). \tag{C.9}$$

The optimum values of the scalars involved are given hereunder

$$\xi_{i\,(\mathrm{opt})} = \frac{B_i}{A_i}, \tag{C.10}$$

$$\theta_{1\,(\mathrm{opt})} = \frac{\left(V_{101}V_{011} - V_{002}V_{110}\right)}{\left(V_{020}V_{002} - V_{011}^2\right)} = \theta_{2\,(\mathrm{opt})}, \tag{C.11}$$

$$\delta_{1\,(\mathrm{opt})} = \frac{\left(V_{110}V_{011} - V_{020}V_{101}\right)}{\left(V_{020}V_{002} - V_{011}^2\right)} = \delta_{2\,(\mathrm{opt})}, \tag{C.12}$$

$$\theta_{4\,(\mathrm{opt})} = \frac{\left(V_{002}V_{110} - V_{101}V_{011}\right)}{\left(V_{020}V_{002} - V_{011}^2\right)} = \theta_{5\,(\mathrm{opt})}, \tag{C.13}$$

$$\delta_{4\,(\mathrm{opt})} = \frac{\left(V_{020}V_{101} - V_{110}V_{011}\right)}{\left(V_{020}V_{002} - V_{011}^2\right)} = \delta_{5\,(\mathrm{opt})}, \tag{C.14}$$

where$A_1 = [1 + V_{200} - 2((V_{020}V_{101}V_{011} - V_{020}V_{002}V_{110} - V_{020}V_{002}V_{101} + V_{002}V_{110}V_{011})/(V_{020}V_{002} - V_{011}^2)) - 2((V_{020}V_{101}^2 + V_{002}V_{110}^2 - 2V_{110}V_{101}V_{011})/(V_{020}V_{002} - V_{011}^2)^2)]$$B_1 = [1 - ((V_{020}V_{101}V_{011} - V_{020}V_{002}V_{110} + V_{002}V_{110}V_{011} - V_{020}V_{002}V_{101})/(V_{020}V_{002} - V_{011}^2))]$$B_4 = [1 + + (((V_{020}V_{002}V_{110} - V_{020}V_{101}V_{011} + V_{020}V_{002}V_{101} - V_{002}V_{110}V_{011}) - (V_{020}V_{101}^2 + V_{002}V_{110}^2 - 2V_{110}V_{101}V_{011}))/2(V_{020}V_{002} - V_{011}^2))]$$B_2 = [1 - (1/2)(V_{020}V_{101}V_{011} + V_{002}V_{110}V_{011} - V_{020}V_{002}V_{110} - V_{020}V_{002}V_{101} - 2V_{110}V_{101}V_{011}/(V_{020}V_{002} - V_{011}^2)) - V_{020}V_{002}(V_{020}V_{101}^2 - 2V_{110}V_{101}V_{011})/(V_{020}V_{002} - V_{011}^2)^2]$$A_4 = [1 + V_{200} + (((V_{020}V_{002}V_{110} - V_{020}V_{101}V_{011} + V_{020}V_{002}V_{101} - V_{002}V_{110}V_{011}) - 2(V_{020}V_{101}^2 + V_{002}V_{110}^2 - 2V_{110}V_{101}V_{011}))/(V_{020}V_{002} - V_{011}^2))]$$A_5 = [1 + V_{200} - (2V_{110}V_{101}V_{011}/(V_{020}V_{002} - V_{011}^2)) + ((3V_{011}^2 - V_{020}V_{002})(V_{020}V_{101}^2 + V_{002}V_{011}^2 - 2V_{110}V_{101}V_{011})/(V_{020}V_{002} - V_{011}^2)^2)]$$B_5 = [1 + V_{011}^2(V_{020}V_{101}^2 + V_{002}V_{011}^2 - 2V_{110}V_{101}V_{011}) - V_{110}V_{101}V_{011}(V_{020}V_{002} - V_{011}^2)/(V_{020}V_{002} - V_{011}^2)^2]$

Similarly, we can obtain the derivation of MSE of the proposed separate estimators.

Data Availability

There are no data associated with this article.

Conflicts of Interest

The authors have no conflicts of interest.

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