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Research Article

On Random Dynamical Systems Generated by White Noise Time Change of Deterministic Dynamical Systems

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In this paper, we apply the random time change by the real white noise to deterministic dynamical systems. We prove that the obtained random dynamical systems are solutions of some stochastic differential equations whenever the deterministic dynamical systems are solutions of ordinary differential equations.

1. Introduction

Random perturbations of deterministic dynamical systems are introduced to model real phenomena, which are usually affected by external fluctuations whose resulting action would be natural to be considered as random. We refer to the monographs [1, 2] (and the references therein) for more details on the subject. A continuous time deterministic dynamical system $S(t,x)$ is solution of a differential equation generated by a vector field f. In general, a random perturbation of S is made either by perturbation of f by a real noise or by adding a white noise term to f (cf. Paragraph 1 below). Following [3, 4], in both cases the resulting process is a random dynamical system and many important deterministic properties are extended to analogous random properties.

Moreover, the idea to consider random time changes for stochastic processes is introduced in [5] and it is extensively studied in many directions (cf. [6–13] for example). However, random time changes for dynamical systems were introduced recently in [14] as new random perturbation of dynamical systems. For a given random time $\tau(t,\omega)$, one can consider the random process $U(t,\omega,x): = S(\tau(t,\omega),x)$, where S is a deterministic dynamical system. Following [14], the aim is to study the properties of $U(t,\omega,x)$ depending on those of the initial dynamical system $S(t,x)$.

In this paper, we consider the random time change by the real white noise of dynamical systems and as application, we investigate the case when these systems are generated by ordinary differential equations.

Let $\Omega = \{\omega\colon \mathbb{R} \longrightarrow \mathbb{R}\colon \omega \text{ is continuous and } \omega(0) = 0\}$ equipped with the compact open topology, let $W\colon \mathbb{R}\times\Omega \longrightarrow \mathbb{R}$ be the real white noise defined by $W(t)(\omega) = \omega(t)$, and let $\theta\colon \mathbb{R}\times\Omega \longrightarrow \Omega$, $(t,\omega)\mapsto\theta(t,\omega)$ defined by $\theta(t,\omega)(s): = \omega(s+t) - \omega(t)$. Then, θ generates a metric dynamical system on Ω (Definitions 1and 2).

Let E be a locally compact space endowed with its Borel σ-algebra B and let $S\colon \mathbb{R}\times E \longrightarrow E$ such that $(E,\mathscr{B},S)$ is a continuous dynamical system on E. Let $\varphi\colon \mathbb{R}\times\Omega\times E \longrightarrow E$ defined by the time change W of S, that is, $\varphi(t,\omega,x) = S(W(t,\omega),x)$. We prove first (Theorem 1) that (θ,φ) is a continuous random dynamical system (Definition 2) on E.

Next, we suppose that E is an open subset of $\mathbb{R}^d$ and $S = (S_1,\ldots,S_d)$ is the solution of the system of ordinary differential equations generated by a $\mathscr{C}^1$-function $f =: (f_1,\ldots,f_d)$, that is,

$$y_k'(t) = f_k(y_1(t),\ldots,y_d(t)), y_k(0) = x_k, 1 \le k \le d. \quad (1)$$

We prove (Theorem 2) that the associated random dynamical system defined in Theorem 1 is the solution $X = : (X_1, \ldots, X_d)$ of the system of stochastic differential equations.

$$dX_k(t) = \frac{1}{2} < \nabla f_k, f > (X(t))dt + f_k(X(t))dW(t); X_k(0) = x_k, 1 \leq k \leq d. \quad (2)$$

Although we considered a particular (but important) random time change, we have established explicit solutions, in terms of the initial solutions, for a class of stochastic differential equations. Notice that in the previous work [14], only expected values of the solutions are investigated, in implicit form, also in terms of the initial solutions.

2. Dynamical and Random Dynamical Systems

For the following classical concepts, we refer essentially to Parts I, II, and IV of [4] (cf. also [15–19]).

Throughout the paper, $\mathbb{R}$ denotes the real line endowed with its Borel σ-algebra R. Moreover, $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^d, d \in \mathbb{N}$ and $\nabla g = ((\partial g/\partial x_1), \ldots, (\partial g/\partial x_d))$ if $g: \mathbb{R}^d \longrightarrow \mathbb{R}$ is a $\mathscr{C}^1$-function.

Definition 1. A dynamical system (DS) is a triplet $(\Gamma, \mathscr{G}, \psi)$ where $(\Gamma, \mathscr{G})$ is a measurable space, $\psi: \mathbb{R} \times \Gamma \longrightarrow \Gamma, (t, y) \mapsto \psi_t y: = \psi(t, y)$ such that

$$\psi(0, y) = y; \psi(s + t, y) = \psi(s, \psi(t, y)), \quad s, t \in \mathbb{R}, y \in \Gamma. \quad (3)$$

If $(t, y) \longrightarrow \psi(t, y)$ is $(\mathscr{R} \otimes \mathscr{G}, \mathscr{G})$ measurable, then the DS $(\Gamma, \mathscr{G}, \psi)$ is said to be measurable.

If Γ is a topological space endowed with its Borel σ-algebra $\mathscr{G}$ and if $(t, y) \longrightarrow \psi(t, y)$ is continuous, then the DS $(\Gamma, \mathscr{G}, \psi)$ is said to be continuous.

In this paper, we consider essentially two different types of DS:

(1) The infinite dimensional case: $(\Gamma, \mathscr{G}, \psi) = (\Omega, \mathscr{F}, \theta)$, where Ω is an infinite dimensional space. A first example is $\Omega: = \{\omega: \mathbb{R} \longrightarrow \mathbb{R}: \omega \text{ is continuous}\}$ equipped with the compact open topology and $\mathscr{F}: = \mathscr{B}(\Omega)$ the associated Borel σ-algebra. We define the translation shift $\theta: \mathbb{R} \times \Omega \longrightarrow \Omega$ by the following equation:

$$\theta(t, \omega)(s) := \omega(s + t), s, t \in \mathbb{R}, \omega \in \Omega. \quad (4)$$

Then, $(\Omega, \mathscr{F}, \theta)$ is a measurable DS, called DS of translations on $\mathbb{R}$. Under additional assumptions, we may define a probability $\mathbf{P}$ on $(\Omega, \mathscr{F})$ such that the DS $(\Omega, \mathscr{F}, \theta)$ becomes a metric DS called usually real noise.

Another important example of infinite dimensional DS, is the Wiener DS on $\mathbb{R}$. It will be considered in the second paragraph.

(2) The finite dimensional case: $(\Gamma, \mathscr{G}, \psi) = (E, \mathscr{B}, S)$ where E is a locally compact space endowed with its Borel σ-algebra $\mathscr{B}$ and S is continuous. In this case, the DS $(E, \mathscr{B}, S)$ is said to be deterministic. The global solutions S of ordinary differential equations on an open subset of $\mathbb{R}^d, d \geq 1$ are the most important examples and they will be treated in the third paragraph.

Definition 2. Let E be a locally compact space endowed with its Borel σ-algebra $\mathscr{B}$.

A measurable random dynamical system (RDS) defined on E consists of two ingredients:

(1) A metric DS, i.e., a measurable DS $(\Omega, \mathscr{F}, \theta)$ endowed with a probability measure $\mathbf{P}$ which is θ-invariant, that is,

$$\mathbf{P}\left(\theta_t^{-1}(F)\right) = \mathbf{P}(F), F \in \mathscr{F}, t \in \mathbb{R}. \quad (5)$$

(2) A cocycle over θ, i.e., a mapping $\varphi: \mathbb{R} \times \Omega \times E \longrightarrow E, (t, \omega, x) \mapsto \varphi(t, \omega, x)$ which is $(\mathscr{R} \otimes \mathscr{F} \otimes \mathscr{B}, \mathscr{B})$ measurable and satisfying

$$\varphi(0, \omega, x) = x, \omega \in \Omega, x \in E. \quad (6)$$

And the cocycle equation:

$$\varphi(s + t, \omega, x) = \varphi(s, \theta_t \omega, \varphi(t, \omega, x)), \quad s, t \in \mathbb{R}, \omega \in \Omega, x \in E. \quad (7)$$

Such a RDS is denoted by $(\Omega, \mathscr{F}, \theta, E, \varphi)$ or simply by (θ, φ) if there is no confusion.

(θ, φ) is said to be continuous, if $(t, x) \longrightarrow \varphi(t, \omega, x)$ is continuous for $\mathbf{P}$-almost every $\omega \in \Omega$.

Let (θ, φ) be a measurable RDS on E. We may associate the skew product $\Phi: \mathbb{R} \times \Omega \times E \longrightarrow \Omega \times E$ defined by the following equation:

$$\Phi_t(\omega, x) := = (\theta(t, \omega), \varphi(t, \omega, x)); t \in \mathbb{R}, x \in E, \omega \in \Omega. \quad (8)$$

Then, $(\Omega \times E, \mathscr{F} \otimes \mathscr{B}, \Phi)$ is a measurable DS on the product space $\Omega \times E$ endowed with the tensor product σ-algebra $\mathscr{F} \otimes \mathscr{B}$.

A first standard class of RDS are solutions of random differential equations (RDE): Let $(\Omega, \mathcal{F}, \theta)$ be a metric DS (for example the translation shift defined by (4), let E be an open subset of $\mathbb{R}^d$, and $h: \Omega \times E \longrightarrow E$ be measurable such that, for each $\omega \in \Omega$, $(t, x) \longrightarrow h(\theta_t \omega, x)$, $(t, x) \longrightarrow h(\theta_t \omega, x)$ is continuous and $x \longrightarrow h(\theta_t \omega, x)$ is locally-Lipschitz. Then, the random differential equation,

$$dX_t(\omega) = h(\theta_t \omega, X_t(\omega))dt; X_0(\omega) = x \in E. \tag{9}$$

admits a unique solution $\varphi(t, \omega, x) = X_t^x(\omega)$ and (θ, φ) is a continuous RDS on E.

A second important class of RDS are solutions of stochastic differential equations (SDE). They will be investigated in the third paragraph.

3. Time Change by Real White Noise

For the following standard notions, we refer essentially to [20], Part IV of [4], and Chapter 2 of [21] (cf. also [17]).

Let $\Omega = \{\omega: \mathbb{R} \longrightarrow \mathbb{R}: \omega \text{ be continuous and } \omega(0) = 0\}$ equipped with the compact open topology, let $\mathcal{F}: = \mathcal{B}(\Omega)$ be the associated Borel σ-algebra, and let $W: \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$, which is defined by the following equation:

$$W(t, \omega) \coloneqq \omega(t), t \in \mathbb{R}, \omega \in \Omega. \tag{10}$$

There exists, by a classical result (Kolmogorov extension theorem), a unique probability measure $\mathbf{P}$ on $(\Omega, \mathcal{F})$ such that the process W has stationary and independent increments and $(W(t, .) - W(s, .))$ has normal distribution with mean 0 and variance $|t - s|$.

Let $\theta: \mathbb{R} \times \Omega \longrightarrow \Omega$, $(t, \omega) \mapsto \theta(t, \omega)$, which is defined by the following equation:

$$\theta(t, \omega)(s) \coloneqq \omega(s + t) - \omega(t), \quad s, t \in \mathbb{R}, \omega \in \Omega. \tag{11}$$

Then, $(\Omega, \mathcal{F}, P, \theta)$ is a metric DS, called the Wiener or Brownian DS. Moreover θ is called Wiener shift on $\mathbb{R}$ and W, which is called real white noise.

The Wiener DS is the appropriate sample space in order to interpret stochastic differential equations (SDE) as RDS.

Remark 1. The main idea of this paper is to investigate the real white noise process $W: \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$ as random time change. Combining equations (7) and (8), we get the following equation:

$$W(s + t, \omega) = W(t, \omega) + W(s, \theta(t, \omega)), \quad s, t \in \mathbb{R}, \omega \in \Omega. \tag{12}$$

According to [20], W is in fact an extension to $\mathbb{R}$ of the white noise process on $[0, \infty)$. Therefore, by equations (7) and (9), W is (the extension of) a random time in the sense of Definition 2.1 of [14].

Now we come to our first result.

Theorem 1. *Let E be a locally compact space endowed with its Borel σ-algebra $\mathcal{B}$ and let $S: \mathbb{R} \times E \longrightarrow E$ such that $(E, \mathcal{B}, S)$ is a continuous DS on E. Let $(\Omega, \mathcal{F}, P, \theta)$ be the metric Wiener DS and let W the associated real white noise. We define $\varphi: \mathbb{R} \times \Omega \times E \longrightarrow E$ by the following equation:*

$$\varphi(t, \omega, x) = S(0(t, \omega), x), \quad t \in \mathbb{R}, x \in E, \omega \in \Omega. \tag{13}$$

Then, (θ, φ) is a continuous RDS on E.

Proof. φ defined by equation (13) is $(\mathcal{R} \otimes \mathcal{F} \otimes \mathcal{B}, \mathcal{B})$ measurable as composition of measurable applications. Indeed, $W: \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$ is $(\mathcal{R} \otimes \mathcal{F}, \mathcal{R})$ measurable by definition of W. Let $I(x): = x, x \in E$, then $(W, I): \mathbb{R} \times \Omega \times E \longrightarrow \mathbb{R} \times E$; $(t, \omega, x) \mapsto (W(t, \omega), x)$ is trivially $(\mathcal{R} \otimes \mathcal{F} \otimes \mathcal{B}, \mathcal{R} \otimes \mathcal{B})$ measurable. Also, $S: \mathbb{R} \times E \longrightarrow E$ is $(\mathcal{R} \otimes \mathcal{B}, \mathcal{B})$ measurable by definition of S and therefore $\varphi = So(W, I)$ is $(\mathcal{R} \otimes \mathcal{F} \otimes \mathcal{B}, \mathcal{B})$ measurable. Similarly, $(t, x) \longrightarrow \varphi(t, \omega, x)$ is continuous for $\mathbf{P}$ for almost every $\omega \in \Omega$ as composition of continuous applications.

Moreover, using equation (13) and the first part of equation (3), we obtain the following equation:

$$\varphi(0, \omega, x) = S((W)(0, \omega), x) = S(0, x) = x, \omega \in \Omega, x \in E. \tag{14}$$

Since $W(0, \omega) = \omega(0) = 0$ by the well definition of Ω.

It remains to prove that φ satisfies the cocycle equation (7). Let $s, t \in \mathbb{R}, x \in E, \omega \in \Omega$. By applying equation (13) to $\overline{\omega}: = \theta(t, \omega)$ and $\overline{x}: = \varphi(t, \omega, x) = S(W(t, \omega), x)$, we obtain the following equation:

$$\varphi(s, \theta(t, \omega), \varphi(t, \omega, x)) = S(W(s, \theta(t, \omega)), S(W(t, \omega), x)). \tag{15}$$

By using equation (12), equation (15) becomes the following equation:

$$\varphi(s, \theta(t, \omega), \varphi(t, \omega, x)) = S(W(s + t, \omega) - W(t, \omega), S(W(t, \omega), x)). \tag{16}$$

Finally, by applying the second part of equation (3) in equation (16), we get the following equation:

$$\begin{aligned}\varphi(s,\theta(t,\omega),\varphi(t,\omega,x)) &= S(W(s+t,\omega)-W(t,\omega)+W(t,\omega),x)\\ &= S(W(s+t,\omega),x)\\ &= \varphi(s+t,\omega,x),\end{aligned} \tag{17}$$

in view of Formula (13). □

Remark 2.

(1) Equation (12) is called helix equation. We refer to [22] for a detailed study of equation (12).

(2) The particular RDS obtained in Theorem 1 is constructed by real white noise time change of a deterministic DS. It seems to be worthwhile to investigate this RDS according to the general theory of RDS as presented in the monograph [4].

(3) In the next paragraph, we deepen the study of this RDS in the particular case when S is generated by an ordinary differential equation.

4. An Application to Ordinary Differential Equations

For the following notions, we refer essentially to Chapters 3, 4, 5, and 7 of [21] (cf. also [17]).

The Wiener DS $(\Omega,\mathscr{F},P,\theta)$ is the appropriate sample space in order to interpret stochastic differential equations (SDE) as RDS. Indeed, the associated white noise W allows to define Itô stochastic integral $\int_0^t f(s,\omega)dW(s)(\omega)$ by putting $W(t)\colon = W(t,.)$ (cf. [21] Chapter 3).

Let E be an open subset of $\mathbb{R}^d$ endowed with its Borel σ-algebra $\mathscr{B}$. A SDE on E is of the form

$$\mathrm{d}X(t) = g(X(t))\mathrm{d}t + f(X(t))\mathrm{d}W(t);\, X(0)=x;\, t\in\mathbb{R}, x\in E, \tag{18}$$

where $f,g\colon E\longrightarrow E$ are locally-Lipschitz functions.

According to [3] Chapter 6, equation (13) admits a unique solution $\varphi\colon \mathbb{R}\times\Omega\times E\longrightarrow E$ and (φ,θ) is a continuous RDS on E.

The following useful result is a particular case of the so-called Itô Formula.

Lemma 1. *Let $\mathbb{R}\longrightarrow\mathbb{R}; r\mapsto u(r)$ be a $\mathscr{C}^2$-function and let $Y(t)\colon = u(W(t))$, then*

$$\mathrm{d}Y(t) = \frac{1}{2}\frac{\mathrm{d}^2u}{\mathrm{d}r^2}(W(t))\mathrm{d}t + \frac{\mathrm{d}u}{\mathrm{d}r}(W(t))\mathrm{d}W(t). \tag{19}$$

Proof. We refer to [21], Theorem 4.1.2 (where W is denoted by B): by taking $X_t = W_t$ and $g(t,r)=u(r)$, formula (4.1.7) of [21] becomes the following equation:

$$\begin{aligned}\mathrm{d}Y(t) &= 0 + \frac{\mathrm{d}u}{\mathrm{d}r}(W(t))\mathrm{d}W(t) + \frac{1}{2}\frac{\mathrm{d}^2u}{\mathrm{d}r^2}(W(t))(\mathrm{d}W(t))^2\\ &= \frac{\mathrm{d}u}{\mathrm{d}r}(W(t))\mathrm{d}W(t) + \frac{1}{2}\frac{\mathrm{d}^2u}{\mathrm{d}r^2}(W(t))\mathrm{d}t,\end{aligned} \tag{20}$$

since $(\mathrm{d}W(t))^2 = \mathrm{d}t$ by formula (4.1.7) of [21].

Next, let E be an open subset of $\mathbb{R}^d, d\geq 1$. For a given $\mathscr{C}^1$-function $f\colon E\longrightarrow E$, we consider the associated autonomous first order ordinary differential equation (ODE)

$$\mathrm{d}y(t) = f(y(t))\mathrm{d}t;\, y(0)=x\in E. \tag{21}$$

The autonomous ODE (21) is said to be generated by f.

We noticed that for $d\geq 2$, the considered ODE is in fact a system. Indeed, if $f=(f_1,\ldots,f_d)$, $y=(y_1,\ldots,y_d)$, and $x=(x_1,\ldots,x_d)$, then equation (21) is equivalent to

$$\mathrm{d}y_k(t) = f_k(y_1(t),\ldots,y_d(t))\mathrm{d}t,\, y_k(0)=x_k, 1\leq k\leq d. \tag{22}$$

Since f is a $\mathscr{C}^1$-function, then equation (21) admits a unique solution for each initial value $x\in E$. The proof of this classical result can be found in chapter 2 of [23]. We suppose that the unique solution of equation (21) is global (cf. [23], chapter 3 for more details). This means that, equation (21) admits a unique solution $S(.,x)\colon \mathbb{R}\longrightarrow E$ for each $x\in E$. In fact, we have defined a system $S\colon \mathbb{R}\times E\longrightarrow E$, $(t,x)\mapsto S(t,x)$ satisfying the following equation:

$$\begin{cases}\dfrac{\partial}{\partial t}S(t,x) = f(S(t,x))\\[2ex] S(0,x) = x\in E\end{cases} \tag{23}$$

It is well known that (E,E,S) is a continuous DS on E (cf. [15, 16, 23]). It is said to be generated by ODE (21).

By applying Theorem 1, we come to the second result of this paper. □

Theorem 2. *Let W be the real white noise and let $S=(S_1,\ldots,S_d)\colon \mathbb{R}\times E\longrightarrow E$ be the solution of the system of ODE's generated by a $\mathscr{C}^1$-function $f=\colon (f_1,\ldots,f_d)$, that is,*

$$\frac{\partial}{\partial t}S_k(t,x) = f_k(S(t,x));\, S_k(0,x)=x_k, 1\leq k\leq d, \tag{24}$$

where $x=(x_1,\ldots,x_d)\in E$. Then, the RDS φ defined by equation (13) is the solution $X=\colon (X_1,\ldots,X_d)$ of the system of SDE's

$$dX_k(t) = \frac{1}{2} < \nabla f_k, f > (X(t))dt + f_k(X(t))dW(t); X_k(0) = x_k, 1 \le k \le d. \quad (25)$$

Proof. Let $1 \le k \le d$. We apply Lemma 1 to the function $u_k(r) = S_k(r, x)$ for $1 \le k \le d, x \in E$. First,

$$\frac{du_k}{dr}(r) = \frac{\partial}{\partial r} S_k(r, x) = f_k(S(r, x)). \quad (26)$$

By derivation of equation (26), we get the following equation by using equations (24) and (26).

$$\begin{aligned} \frac{d^2 u_k}{dr^2}(r) &= \frac{\partial}{\partial r} f_k(S(r, x)) \\ &= \frac{\partial}{\partial r} f_k(S_1(r, x), \ldots, S_d(r, x)) \\ &= \sum_{i=1}^{i=d} \frac{\partial}{\partial x_i} f_k(S(r, x)) \frac{\partial}{\partial r} S_i(r, x) \\ &= \sum_{i=1}^{i=d} \frac{\partial}{\partial x_i} f_k(S(r, x)) f_i(S(r, x)). \end{aligned} \quad (27)$$

Hence,

$$\frac{d^2 u_k}{dr^2}(r) = < \nabla f_k, f > (S(r, x)). \quad (28)$$

Let $X_k(t)(\omega)x: = S_k(W(t), x)$. Notice first that from equation (24)

$$X_k(0) = S_k(W(0), x) = S_k(0, x) = x_k. \quad (29)$$

Moreover, we have the following equation:

$$X(t) := (X_1(t), \ldots, X_d(t)) = S(W(t)). \quad (30)$$

By using equations (20)–(22), the Itô formula (19) gives the following equation:

$$\begin{aligned} dX_k(t) &= \frac{1}{2}(\langle \nabla f_k, f \rangle) S(W(t))dt + f_k S(W(t))dW(t) \\ &= \frac{1}{2}(\langle \nabla f_k, f \rangle)(X(t))dt + f_k(X(t))dW(t). \end{aligned} \quad (31)$$

This completes the proof. □

Remark 3. In Theorem 2, we have associated a SDE to a given ODE by the random time change W. Our approach is completely different from the classical idea, mentioned in the introduction, which consists of adding a stochastic term (say $g(X(t))dW(t)$) to an ODE (of the form $dX(t) = f(X(t))dt$) in order to obtain the SDE $dX(t) = f(X(t))dt + g(X(t))dW(t)$.

Example 1. Recall that $(\Omega, \mathcal{F}, P, \theta)$ is the Wiener DS and W is the associated white noise. For simplicity, we suppose that $E: = \mathbb{R}^d$.

For the one-dimensional case, Theorem 2 reads as follows: Let $S: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be the DS solution of the ODE generated by a $\mathscr{C}^1$-function $f: \mathbb{R} \longrightarrow \mathbb{R}$. Then, $(t, \omega, x) \mapsto \varphi(t, \omega, x): = S(W(t, \omega), x)$ is the solution X of the SDE

$$dX(t) = \left(\frac{1}{2}\right)(f' f)(X(t))dt + f(X(t))dW(t); X(0) = x \in \mathbb{R}. \quad (32)$$

For example, if $f(x) = x(1 - x), x \in \mathbb{R}$, then $S(t, x) = x(x + (1 - x)e^{-t})^{-1}; t, x \in \mathbb{R}$. Hence, the SDE

$$dX(t) = \left(\frac{1}{2}\right) X(t)\left(1 - 3X(t) + 2X^2(t)\right)dt + X(t)(1 - X(t))dW(t); X(0) = x \in \mathbb{R}. \quad (33)$$

Admits a unique solution given by $\varphi(t, \omega, x) = x(x + (1 - x)e^{-W(t,\omega)})^{-1}$.

Example 2. For the two-dimensional case, consider the system of ODE's

$$\begin{cases} dy_1 = f_1(y_1, y_2)dt \\ dy_2 = f_2(y_1, y_2)dt \\ (y_1(0), y_2(0)) = (x_1, x_2) = x \in \mathbb{R}^2 \end{cases}, \quad (34)$$

where $f: = (f_1, f_2): \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a $\mathscr{C}^1$-function and let $S: \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ the DS solution of this system. Then, $\varphi =$

$(\varphi_1, \varphi_2)\colon \mathbb{R}\times\Omega\times\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $\varphi(t,\omega,x)\colon = S(W(t)(\omega),x)$ is the solution $X\colon = (X_1, X_2)$ of the system of SDE's

$$\begin{cases} \mathrm{d}X_1(t) = \left(\frac{1}{2}\right)\left(\frac{\partial f_1}{\partial x_1}f_1 + \frac{\partial f_1}{\partial x_2}f_2\right)(X(t))\mathrm{d}t + f_1(X(t))\mathrm{d}W(t) \\ \mathrm{d}X_2(t) = \left(\frac{1}{2}\right)\left(\frac{\partial f_2}{\partial x_1}f_1 + \frac{\partial f_2}{\partial x_2}f_2\right)(X(t))\mathrm{d}t + f_2(X(t))\mathrm{d}W(t) \end{cases}. \tag{35}$$

With the initial condition $(X_1(0), X_2(0)) = (x_1, x_2) = x \in \mathbb{R}^2$.

Example 3. We illustrate the three-dimensional case by considering an example on $\mathbb{R}^3$. We consider the system of ODEs

$$\begin{cases} \mathrm{d}y_1 = y_2 y_3 \mathrm{d}t \\ \mathrm{d}y_2 = (y_1 + y_3)\mathrm{d}t. \\ \mathrm{d}y_3 = y_1^2 \mathrm{d}t \end{cases} \tag{36}$$

Let $S\colon \mathbb{R}\times\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the DS solution of this system. Then, $\varphi = (\varphi_1, \varphi_2, \varphi_3)\colon \mathbb{R}\times\Omega\times\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $\varphi(t,\omega,x)\colon = S(W(t)(\omega),x)$ is the solution $X\colon = (X_1, X_2, X_3)$ of the system of SDE's

$$\begin{cases} \mathrm{d}X_1(t) = \left(\frac{1}{2}\right)\left(X_1(t)X_3(t) + X_3^2(t) + X_2(t)X_1^2(t)\right)\mathrm{d}t + X_2(t)X_3(t)\mathrm{d}W(t) \\ \mathrm{d}X_2(t) = \left(\frac{1}{2}\right)\left(X_2(t)X_3(t) + X_1^2(t)\right)\mathrm{d}t + \left(X_1(t) + X_3(t)\right)\mathrm{d}W(t) \\ \mathrm{d}X_3(t) = X_1(t)X_2(t))X_3(t)\mathrm{d}t + X_1^2(t)\mathrm{d}W(t) \end{cases}. \tag{37}$$

With the initial condition $\{X_1(0), X_2(0), X_3(0) = (x_1, x_2, x_3) = x \in \mathbb{R}^3$.

Remark 4. Theorem 2 can be read as follows: we consider the SDE

$$\mathrm{d}X(t) = g(X(t))\mathrm{d}t + f(X(t))\mathrm{d}W(t); X(0) = x; t \in \mathbb{R}, \quad x \in E, \tag{38}$$

where $f, g\colon E \longrightarrow E$ are two $\mathscr{C}^1$-functions. We consider the ODE

$$\mathrm{d}y(t) = f(y(t))\mathrm{d}t; y(0) = x \in E. \tag{39}$$

And let $S(t,x); t \in \mathbb{R}, x \in E$ be the solution of the ODE (39). Then, $\varphi(t,\omega,x)\colon = S(W(t,\omega),x); t \in \mathbb{R}, \omega \in \Omega, x \in E$ is the solution of the SDE (38) if and only if

$$g_k = \langle \nabla f_k, f \rangle; 1 \le k \le d, \tag{40}$$

where $f = \colon (f_1, \ldots, f_d)$ and $g = \colon (g_1, \ldots, g_d)$.

Hence, we have reduced the resolution of a class of stochastic differential equations to the resolution of the associated ordinary differential equations.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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