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Approximate potentials with applications to strongly nonlinear oscillators with slowly varying parameters

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Abstract. A method of approximate potential is presented for the study of certain kinds of strongly nonlinear oscillators. This method is to express the potential for an oscillatory system by a polynomial of degree four such that the leading approximation may be derived in terms of elliptic functions. The advantage of present method is that it is valid for relatively large oscillations. As an application, the elapsed time of periodic motion of a strongly nonlinear oscillator with slowly varying parameters is studied in detail. Comparisons are made with other methods to assess the accuracy of the present method.

Keywords: Strongly nonlinear oscillator, approximate potential, slowly varying parameter

1. Introduction

In control engineering and oscillatory problems, we often meet with the following strongly nonlinear oscillator

$$\frac{d^2y}{dt^2} + \varepsilon k(y,\tilde{t})\frac{dy}{dt} + g(y,\tilde{t}) = 0 \tag{1}$$

where $\tilde{t} = \varepsilon t (0 < \varepsilon \ll 1)$ is the slow scale. Methods of multiple scales and generalized KBM are effective to deal with such systems [1–4]. However, these methods are based on the fact that the reduced equation $(\varepsilon = 0)$, although nonlinear, has a known solution in terms of special functions. Jacobian elliptic functions are used when $g(y,\tilde{t})$ is a polynomial of degree two or three with respect to y. For generally nonlinear functions $g(y,\tilde{t})$, Taylor expansions are often used to approximate them but they are effective only for small amplitudes [5]. Many efforts have been done to overcome the difficulty of nonlinearity, such as Fourier series [6], stroboscopic method [7], energy method [8], generalized harmonic function [9], perturbation-incremental method [10]. Approximate potential method was first proposed by the author Li in [11] to deal with a generalized pendulum equation resulted from the free electron laser (FEL). In [11] the potential for the nonlinear oscillator is expressed by a polynomial of degree three such that the leading approximation is expressible in terms of elliptic functions. This paper is to explore this idea for some other kinds of oscillatory systems, in which the potentials can be approximated by polynomials of degree four. Compared with Taylor expansions, the advantage of present method is that it is valid for relatively large oscillations. The present method is illustrated with some examples and an application to a strongly nonlinear oscillator with slowly varying parameters. Comparisons are also made with the numerical method and Taylor expansions method to show the efficiency of the present method.

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2. The basic idea of approximate potential method

We use the reduced equation of Eq. (1)

$$\frac{d^2y}{dt^2} + g(y) = 0 \tag{2}$$

to illustrate the main idea of approximate potential. The idea comes from the analysis of the cubic nonlinear oscillator

$$\frac{d^2y}{dt^2} + ay + by^3 = 0 \tag{3}$$

Its energy integral is

$$\frac{1}{2}(\frac{dy}{dt})^2 + V(y,a,b) = 0 \tag{4}$$

where

$$V(y,a,b) = \frac{1}{2}ay^2 + \frac{1}{4}by^4$$

is the potential. Different possible combinations of the signs of a and b will determine the case for which Eq. (3) can have periodic solutions and can be expressed in terms of elliptic functions.

(1) If $a > 0$ and $b > 0$, the potential V is "U–Shaped" and there are periodic solutions around oscillatory center $y = 0$.
(2) If $a > 0$ and $b < 0$, the potential V is "M–Shaped" and there are periodic solutions around oscillatory center $y = 0$.
(3) If $a < 0$ and $b > 0$, the potential V is "W–Shaped" and there are two families of periodic solutions centered about $y = \pm\sqrt{-\frac{a}{b}}$.
(4) If $a < 0$ and $b < 0$, the potential V is "∩–Shaped" and therefore, none of the solutions are periodic.

If the potential of Eq. (2) $V = \int_0^y g(u)du$ is "U–Shaped", "M–Shaped" or "W–Shaped", we may fit a polynomial of degree four to the potential such that the periodic solution is expressible in terms of elliptic functions.

Example 1. Consider the following nonlinear differential equation

$$\frac{d^2y}{dt^2} + a\sin y - \frac{2}{\pi}y = 0 \tag{5}$$

The energy integral is

$$\frac{1}{2}(\frac{dy}{dt})^2 + V(y) = 0 \tag{6}$$

where

$$V(y) = -a\cos y - \frac{1}{\pi}y^2 + a \tag{7}$$

is the potential (for simplicity we chose $V(0) = 0$). When $a \leqslant 1$, $V(y)$ has a minimum point at $y = 0$ and two maximum points at $y = \pm y_s$. $0 < y_s \leqslant \frac{\pi}{2}$ With this potential, the integral of Eq. (6) cannot be expressed in terms of any elemental or known functions. Approximate approaches must be used. The potential is "M–Shaped", so we may seek a polynomial of degree four to approximate it. The approximate potential is denoted by

$$\overline{V}(y) = by^2 + cy^4$$

where the coefficients are chosen such that

$$\overline{V} = V \text{ and } \overline{V}'_y = 0 \text{ at } y = 0 \text{ and } y = y_s$$

For $a = 1$, we can get

$$\overline{V} = 0.17395y^2 - 0.0352496y^4 \tag{8}$$

If the error between V and $\overline{V}$ in the concerned interval $[y_1, y_2]$ is defined as

$$E = \frac{\int_{y_1}^{y_2} |V - \overline{V}|dy}{\int_{y_1}^{y_2} |V|dy}$$

then the error between Eqs (7) and (8) in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is 1.4%. Substituting $\overline{V}$ for V in Eq. (6) and integrating it, we can obtain the approximate solution of Eq. (5) in terms of elliptic functions. Comparisons of the potential and solution and their approximations are shown in Figs 1 and 2 when $a = 1$ and $y(0) = \frac{2\pi}{5}$, $y'(0) = 0$. In this paper, numerical results are obtained by using Software *Mathematica*.

Example 2. Consider a mass slides on a smooth surface while restrained by a linear spring modeled as [5]

$$\frac{d^2y}{dt^2} + 2y - \frac{2ay}{\sqrt{1+y^2}} = 0 \tag{9}$$

The energy integral is of form Eq. (6) and the potential is

$$V(y) = y^2 - 2a\sqrt{1+y^2} + 2a \tag{10}$$

(1) $0 < a < 1$

For this case, the potential has only one minimum point at $y = 0$. The potential is "U–Shaped", so we may seek a polynomial of the form

$$\overline{V}(y) = by^2 + cy^4$$

to fit the potential V. The coefficients are chosen such that

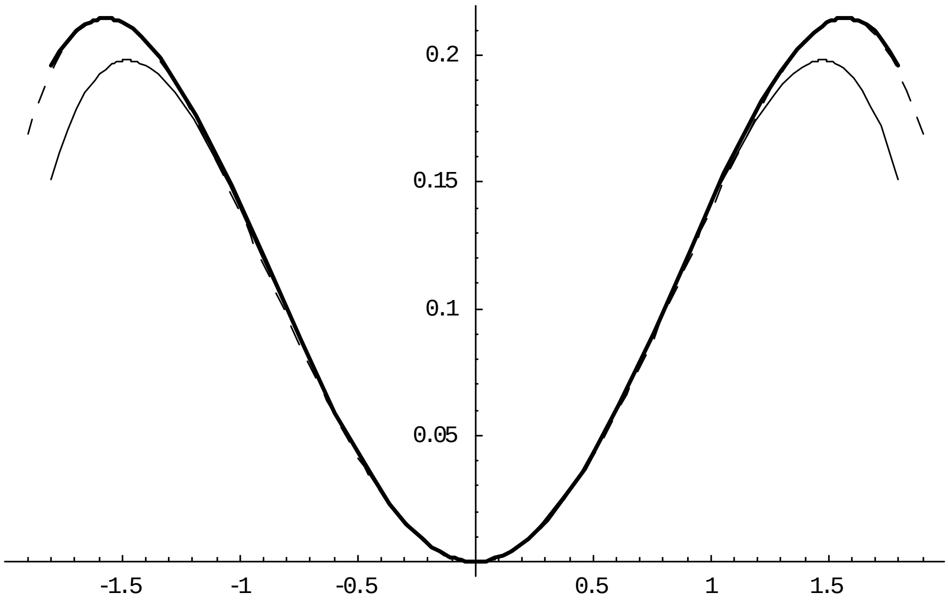

Fig. 1. Potential and approximations of (7) with $a = 1$. Thick curve for original potential (7), dashing curve for approximate potential (8), thin curve fourth-order Taylor expansions of (7).

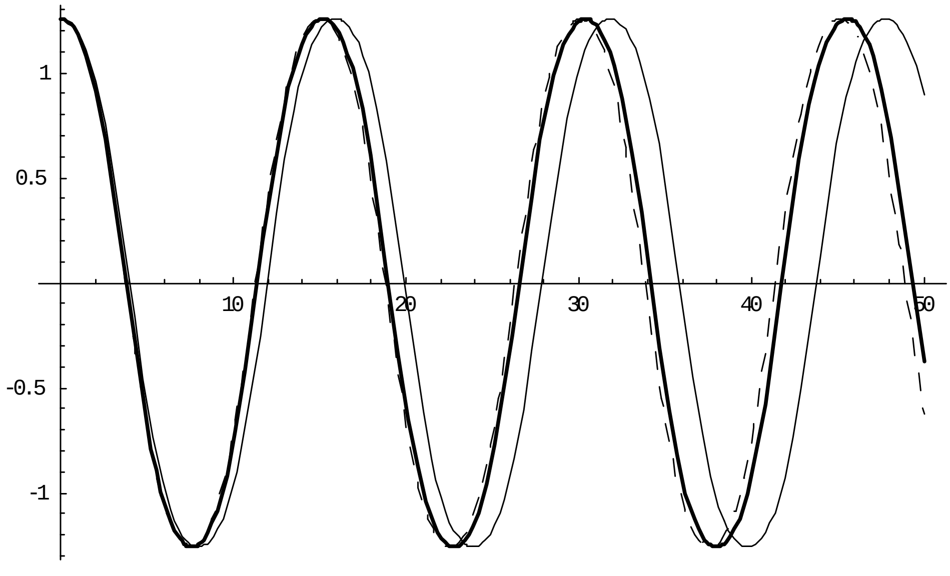

Fig. 2. Solution and approximations of (5) with $y(0) = \frac{2\pi}{5}$, $y'(0) = 0$ and $a = 1$. Thick curve for numerical solution, dashing curve for the present method, thin curve solution with fourth-order Taylor expansions.

$$\overline{V} = V \text{ at } y = 0, y = 1 \text{ and } y = 2.5$$

$$\overline{V}'_y = 0 \text{ at } y = 0$$

In this case, the fitting points 1, 2.5 can be chosen according to concerned range of amplitude. For $a = 0.5$, we have

$$\overline{V} = 0.558471y^2 + 0.0273145y^4 \tag{11}$$

The error between Eqs (10) and (11) in the interval $[-2.5, 2.5]$ is 2.2%. Substituting $\overline{V}$ for V in Eq. (6) and integrating it, we can obtain the approximate solution of Eq. (9) in terms of elliptic functions. Comparisons of the potential and solution and their approximations are shown in Figs 3 and 4 when $a = 0.5$ and $y(0) = \frac{2\pi}{3}, y'(0) = 0$.

(2) $a > 1$

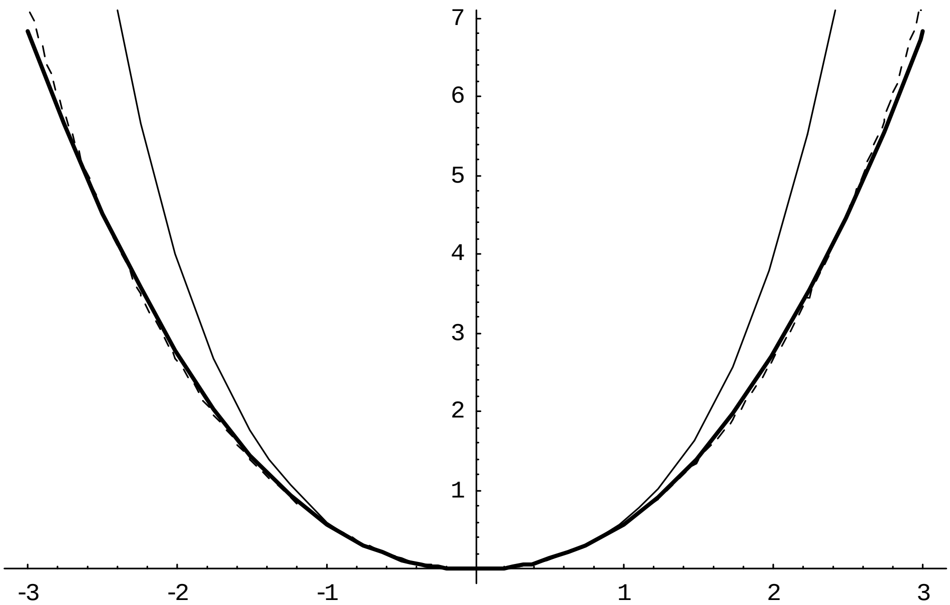

Fig. 3. Potential and approximations of (10) with $a = 0.5$. Thick curve for original potential (10), dashing curve for approximate potential (11), thin curve fourth-order Taylor expansions of (10).

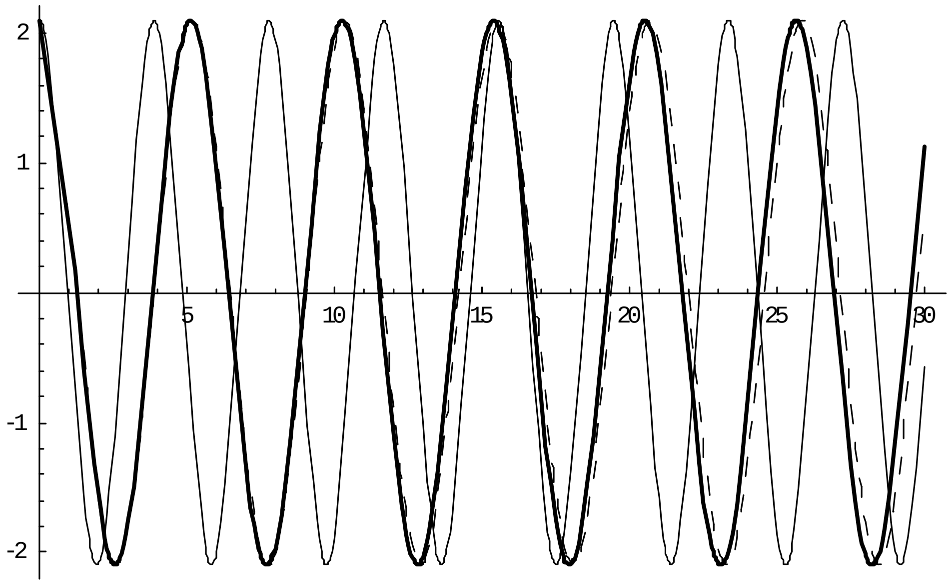

Fig. 4. Solution and approximations of (9) with and $y(0) = \frac{2\pi}{3}, y'(0) = 0$ and $a = 0.5$. Thick curve for numerical solution, dashing curve for the present method, thin curve solution with fourth-order Taylor expansions.

For this case, the potential has one minimum point at $y = 0$ and two maximum points at$y = \pm\sqrt{a^2 - 1}$. The potential is "W–Shaped", so we may seek a polynomial of the form

$$\overline{V}(y) = by^2 + cy^4$$

to fit the potential V. The coefficients are chosen such that

$$\overline{V} = V \text{ and } \overline{V}_y^{'} = 0 \text{ at } y = 0 \text{ and }$$

$$y = \sqrt{a^2 - 1}$$

For $a = 1.1$, we can get

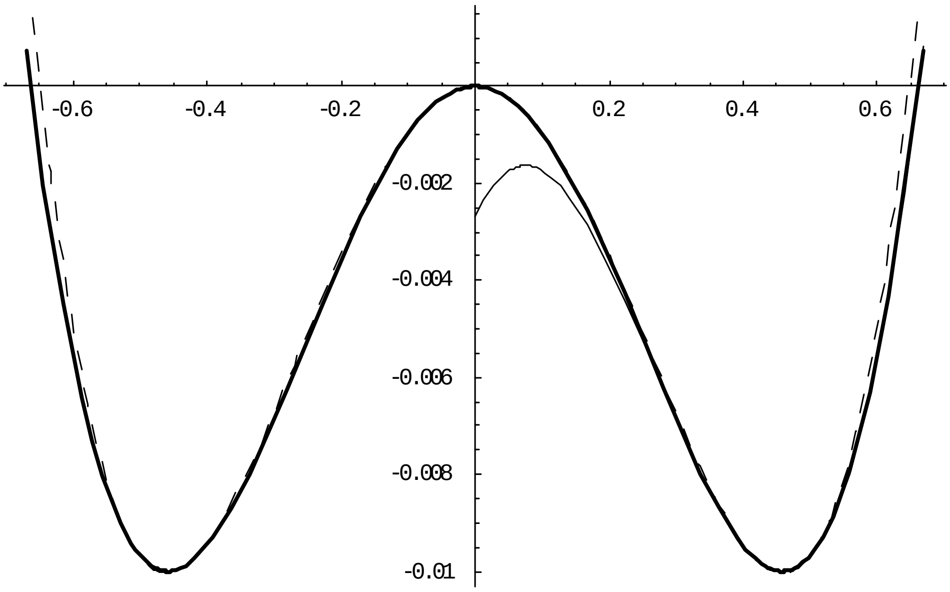

Fig. 5. Potential and approximations of (10) with $a = 1.1$. Thick curve for original potential (10), dashing curve for approximate potential (12), thin curve fourth-order Taylor expansions of (10) at oscillatory center $y = \sqrt{0.21}$.

$$\overline{V} = -0.0952381y^2 + 0.226757y^4 \tag{12}$$

The error between Eqs (10) and (12) in the interval $[0, 0.66]$ is 3.8%. Substituting $\overline{V}$ for V in Eq. (6) and integrating it, we can obtain the approximate solution of Eq. (9) in terms of elliptic functions. Comparisons of the potential and solution and their approximations are shown in Figs 5 and 6 when $a = 1.1$ and $y(0) = 0.1, y'(0) = 0$.

3. Application to strongly nonlinear oscillators with slowly varying parameters

We now apply the results summarized in Section 2 to a particle on a rotating circle or a simple pendulum attached to a rotating rigid frame [5]

$$\frac{d^2y}{dt^2} + \varepsilon k(y, \tilde{t})\frac{dy}{dt} + (a(\tilde{t}) - \cos y)\sin y = 0 \tag{13}$$

where $\tilde{t} = \varepsilon t (0 < \varepsilon \ll 1)$ is the slow scale. The fast scale t^+, following Kuzmak [1], is defined as

$$\frac{dt^+}{dt} = \omega(\tilde{t})$$

with an unknown $\omega(t)$ to be determined by the periodicity of the solution of Eq. (13). Suppose that the solution of Eq. (13) can be developed in the multiple scales form

$$\begin{aligned} y(t, \varepsilon) &= y_0(t^+, \tilde{t}) + \varepsilon y_1(t^+, \tilde{t}) \\ &+ \varepsilon^2 y_2(t^+, \tilde{t}) + \ldots \end{aligned} \tag{14}$$

where $y_0, y_1, \ldots$ must be periodic functions of t^+. Substituting Eq. (14) into Eq. (13) and equating powers of ε gives the leading order equation

$$\omega^2(\tilde{t})\frac{\partial^2 y_0}{\partial t^{+2}} + (a(\tilde{t}) - \cos y_0)\sin y_0 = 0 \tag{15}$$

Multiplying Eq. (15) by $\frac{\partial y_0}{\partial t^+}$ and integrating it with respect to t^+, we obtain the energy integral

$$\frac{\omega^2(\tilde{t})}{2}(\frac{\partial y_0}{\partial t^+})^2 + V(y_0) = E_0(\tilde{t}) \tag{16}$$

where

$$V(y_0) = a(\tilde{t}) - \frac{1}{2} - (a(\tilde{t}) - \frac{1}{2}\cos y_0)\cos y_0 \tag{17}$$

is the potential and $E_0(\tilde{t})$ is the slowly varying energy of the system. When $a(\tilde{t}) \geqslant 1$, $V(y_0$ has a minimum point at $y_0 = 0$ and two maximum points at $y_0 = \pm\pi$. With this potential, the leading approximation Eq. (15) is difficult to be expressed in terms of known functions. From Fig. 7 we see that the fourth-order Taylor expansions of the potential Eq. (17) with $a(\tilde{t}) = 3$ has large unacceptable errors when the amplitude is not small. The worse is that the Taylor expansions cannot describe the character of the potential well. Instead, we construct a fourth-order polynomial to match the minimum and maximum values of Eq. (17). This potential

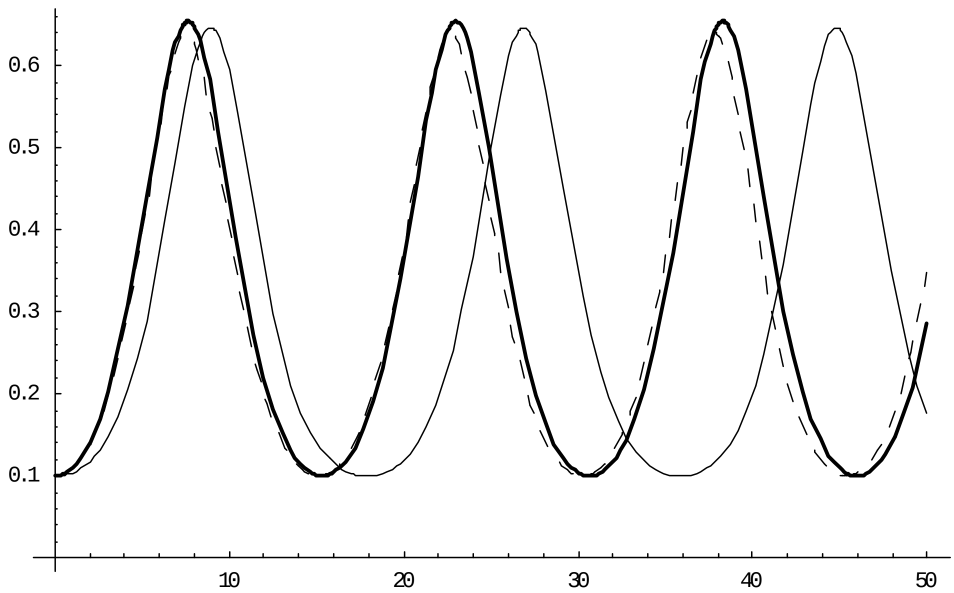

Fig. 6. Solution and approximations of (9) with $y(0) = 0.1, y'(0) = 0$ and $a = 1.1$. Thick curve for numerical solution, dashing curve for the present method, thin curve solution with fourth-order Taylor expansions.

is denoted by

$$\overline{V}(y) = b(\tilde{t})y^2 + c(\tilde{t})y^4$$

where the coefficients are chosen such

$$\overline{V} = V \text{ and } \overline{V}_y^{'} = 0 \text{ at } y = 0 \text{ and } y = \pi$$

Then, we have

$$\overline{V}(y) = \frac{4}{\pi^2}a(\tilde{t})y^2 - \frac{2}{\pi^4}a(\tilde{t})y^4 \tag{18}$$

Substituting $\overline{V}$ for V in Eq. (16) and integrating it, we can obtain y_0 in terms of elliptic functions (see [12] Section 3.6 for details)

$$y_0 = \sqrt{\frac{2\pi^2 v}{1+v}} sn[K(v)\varphi, v] \tag{19}$$

where $\varphi = t^+ + \varphi_0$, and $K(v)$ is the complete elliptic integral of the first kind associated with the modulus v. The equation governing v is in the form (see [6] or [11] for details)

$$\frac{L^2(v)v^2}{(1+v)^3} = \frac{c^2}{32\pi^2 a(\tilde{t})} \exp(-2\int_0^{\tilde{t}} k(0,\tau)d\tau) \tag{20}$$

where constant c can be determined by initial values of the system, and

Table 1
Escaping time from potential well of system (22)

ε	Asymptotic results	Numerical results	errors (%)
0.1	11.195	11.605	3.5
0.01	111.95	114.70	2.4
0.001	1119.5	1120.7	0.1
0.0001	11195	11204	0.08

$$L(v) = \int_0^K cn^2(u,v)dn^2(u,v)du = \frac{1}{3v}[(1+v) E(v) - (1-v)K(v)]$$

Here, $E(v)$ is the complete elliptic integral of the second kind associated with the modulus $\sqrt{v}$.

It can be seen from potential Eq. (17) that the potential well is "M–Shaped". In this case, if there exists the effect of a negative damping in system, the amplitude will be gradually increasing, and the motion will be forced out of the potential well and cease to be periodic. From Eq. (19), the amplitude increasing implies the modulus v increasing. Once v reaches 1, y_0 is no longer a periodic function of φ. This can be used to determine the elapsed time when the motion ceases to be periodic. Denote the elapsed time as T. From Eq. (20) we have

$$\frac{c^2}{8\pi^2 a(T)} \exp(-2\int_0^{\varepsilon T} k(0,\tau)d\tau) = \frac{2}{9} \tag{21}$$

Here, the fact that $L(v) \to \frac{2}{3}$ as $v \to 1$ has been used.

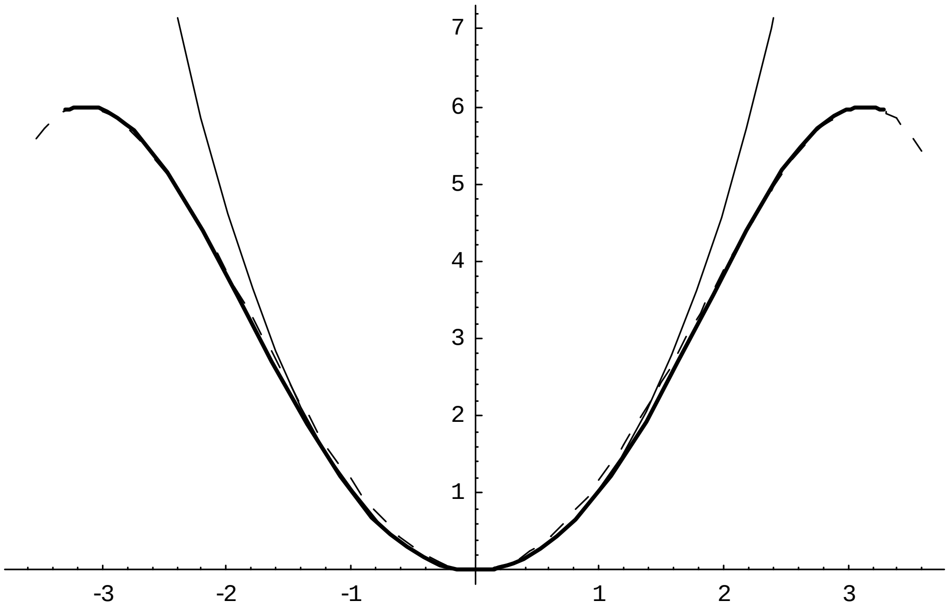

Fig. 7. Potential and approximations of (17) with $a(\tilde{t}) = 3$. Thick curve for original potential (17), dashing curve for approximate potential (18), thin curve fourth-order Taylor expansions of (17).

Example 3. Consider the nonlinear system Eq. (13) with a small negative damping

$$\frac{d^2y}{dt^2} - \varepsilon\frac{dy}{dt} + (3 + \varepsilon t - \cos y)\sin y = 0 \qquad (22)$$

$$y(0) = \frac{\pi}{2}, \;\; y'(0) = 0 \qquad (23)$$

Substituting initial values Eq. (23) into Eq. (19) and Eq. (20), we obtain

$$\varphi_0 = 1 \;\; v = \frac{1}{7}, \;\; c^2 = 7.702575$$

The elapsed time T can be solved from Eq. (21)

$$T = \frac{1.1195}{\varepsilon} \qquad (24)$$

Comparisons of asymptotic results Eq. (24) and numerical results of Eq. (22) are shown in Table 1.

4. Conclusions

(1) The method of approximate potential presented in this paper is effective for certain strongly nonlinear oscillators whose potentials are "M–Shaped", "U–Shaped" and "W–Shaped". The method works not only for small oscillations but also for relatively large oscillations.

(2) The comparisons show that the results of present method are in good agreement with the numerical results, while the method of Taylor expansions have large errors when the amplitudes are not small.

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