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Dynamic analysis of a Timoshenko beam subjected to moving concentrated forces using the finite element method

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Abstract. This paper presents finite element formulations for an elastic Timoshenko beam subjected to moving concentrated forces. The results obtained by the present method are compared with those obtained by the assumed mode method published in the existing literature to verify the correctness of the present method. The present method can analyze the dynamic response for a Timoshenko or Bernoulli-Euler beam with various boundary conditions subjected to moving concentrated forces. Numerical results show that the present method is more effective than the discrete element technique published in the existing literature for investigating the dynamic problem of Timoshenko beam.

Keywords: Timoshenko beam, finite element method, moving concentrated force, dynamic analysis, Bernoulli-Euler beam, equation of motion

1. Introduction

The analysis of moving loads on an elastic structure has been a topic of interest for well over a century. Two kinds of methods, i.e., analytical and numerical methods, are widely used to tackle the problem. Based on the Bernoulli-Euler beam theory, many analytical methods have been proposed to solve simple moving load problems [7, 18,20]. As analytical methods are often limited to simple moving load problems, many researchers have resorted to various numerical methods. Ross [19] investigated the problem of a viscoelastic Timoshenko beam subjected to a transversely applied step-loading was solved using the Laplace transform method. Katz et al. [11] studied the dynamic behavior of a rotating shaft subjected to a moving load with constant velocity using the modal analysis method and an integral transformation method. In their paper, the Bernoulli-Euler, Rayleigh and Timoshenko beam theories were used to model the rotating shaft. Akin and Mofid [1] presented an analytical-numerical method for determining the dynamic behavior of Bernoulli-Euler beams with different boundary conditions and carrying a moving mass. Han and Zu [8] investigated the dynamic behavior of a spinning Timoshenko beam subjected to a constant moving load using a modal expansion technique. Lee [12,13] reported the dynamic responses of a rotating shaft subjected to axial force and moving loads and of a Timoshenko beam subjected to a moving mass using the assumed mode method (AMM).

Although varying positions of the moving loads need to some special considerations, the finite element method is especially powerful due to its versatility in the spatial discretization. Yoshida and Weaver [27] first applied the finite element method to the moving load problem. It has since been used by many other investigators [2,4,6,9,10,14–17,

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21–23]. References [2,4,6,9,10,14–17,21–23] are limited to the dynamic analysis of moving loads on structures based on the Bernoulli-Euler beam theory. The existing literatures about the dynamic problem of moving loads on the Timoshenko beam using the finite element method are small. Yavari et al. [24] analyzed the dynamic response of Timoshenko beams under a moving mass using the discrete element technique (DET).

In the present paper, finite element formulations for an elastic beam subjected to moving concentrated forces will be presented. The beam is discretized into a number of simple elements with four degrees of freedom each. The shape functions, which are presented by Yokoyama [25,26] of the vertical displacement and the bending rotation for a Timoshenko beam element, are used in this paper. The equation of motion in matrix form for a Timoshenko beam element with four degrees of freedom acted upon by moving concentrated forces is derived from the principle of virtual work. By assembling element matrices and element nodal vectors, respectively, the global equation of motion for a Timoshenko beam subjected to moving concentrated forces are obtained. The problem is solved by direct integration using Wilson θ method or similar methods [3], to obtain the dynamic response of a Timoshenko beam. The present formulation can analyze the dynamic response of a Timoshenko beam with various boundary conditions subjected to moving concentrated forces, and also investigate the dynamic problem of an elastic Bernoulli-Euler beam.

2. Theory and formulation

2.1. The model of a Timoshenko beam element

A simply supported Timoshenko beam subjected to a series of moving concentrated forces with speed v along the beam is shown in Fig. 1. A co-ordinate system is assumed to be fixed in the inertial frame, with the x-axis parallel to the undeformed longitudinal axis of the beam and the y-axis pointing vertically downward in the same direction as the gravitational acceleration g. It is assumed that the downward displacement of the Timoshenko beam is taken as positive and that it is measured with reference to its vertical static equilibrium position.

The Timoshenko beam is discretized into a number of simple elements with equal length each. Figure 2 shows a Timoshenko beam element of length l with a few concentrated forces running on it. It is assumed that the total number of the concentrated forces running on the beam element is n_f. The beam element consists of two nodes i and j; each node has two degrees of freedom, i.e., vertical displacement y^e and bending rotation (or slope) θ^e. The vertical displacement y^e and bending rotation (or slope) θ^e of an arbitrary point on the beam element can be expressed [26]

$$y^e = \left[N_{y1}\ N_{y2}\ N_{y3}\ N_{y4} \right] \begin{Bmatrix} y_i^e \\ \theta_i^e \\ y_j^e \\ \theta_j^e \end{Bmatrix} = [N_y]\{q\}^e \tag{1}$$

$$\theta^e = \left[N_{\theta 1}\ N_{\theta 2}\ N_{\theta 3}\ N_{\theta 4} \right] \begin{Bmatrix} y_i^e \\ \theta_i^e \\ y_j^e \\ \theta_j^e \end{Bmatrix} = [N_\theta]\{q\}^e \tag{2}$$

Here $\{q\}^e$ is the element nodal displacement vector, and the shape functions (or the interpolation functions) are written as [26]

$$N_{y1} = [1 - 3\xi^2/l^2 + 2\xi^3/l^3 + (1 - \xi/l)\Phi]/(1 + \Phi)$$

$$N_{y2} = [\xi - 2\xi^2/l + \xi^3/l^2 + (\xi - \xi^2/l)\Phi/2]/(1 + \Phi)$$

$$N_{y3} = [3\xi^2/l^2 - 2\xi^3/l^3 + \Phi\xi/l]/(1 + \Phi)$$

$$N_{y4} = [-\xi^2/l + \xi^3/l^2 - (\xi - \xi^2/l)\Phi/2]/(1 + \Phi)$$

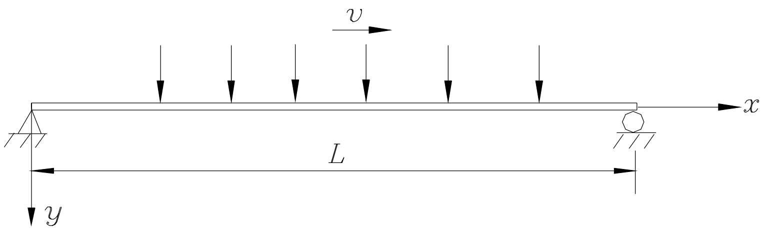

Fig. 1. A simply supported Timoshenko beam subjected to a series of moving concentrated forces.

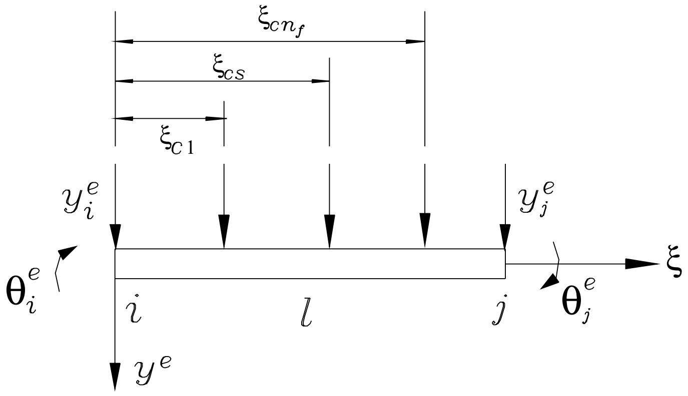

Fig. 2. Model of a Timoshenko beam element with a few moving concentrated forces on it.

$$N_{\theta 1} = 6[-\xi/l^2 + \xi^2/l^3]/(1+\Phi)$$

$$N_{\theta 2} = [1 - 4\xi/l + 3\xi^2/l^2 + (1-\xi/l)\Phi]/(1+\Phi)$$

$$N_{\theta 3} = 6[\xi/l^2 - \xi^2/l^3]/(1+\Phi)$$

$$N_{\theta 4} = [-2\xi/l + 3\xi^2/l^2 + \Phi\xi/l]/(1+\Phi)$$

where ξ is the local coordinate along the axis of the beam element; $\Phi = 12EI/(k_s GAl^2)$ = the shear deformation parameter; E Young's modulus; I the second moment of area; k_s the shear coefficient depending on the shape of the cross-section; G the shear modulus; A the cross-sectional area; and l the length of beam element.

2.2. *The virtual work of a Timoshenko beam element under a few concentrated forces*

Figure 2 shows a Timoshenko beam element of length l with a few concentrated forces running on it. The virtual work of this beam element consists of the internal virtual work δW_I^e and the external virtual work δW_E^e of this beam element. It is assumed that the damping effect of the beam is neglected, the internal virtual work δW_I^e of this beam element can be written as

$$\delta W_I^e = \int_0^l EI\left(\frac{\partial\theta^e}{\partial\xi}\right)\cdot\delta\left(\frac{\partial\theta^e}{\partial\xi}\right)\mathrm{d}\xi + \int_0^l k_s GA\left(\frac{\partial y^e}{\partial\xi} - \theta^e\right)\cdot\delta\left(\frac{\partial y^e}{\partial\xi} - \theta^e\right)\mathrm{d}\xi \tag{3}$$

The external virtual work δW_E^e of this beam element can be expressed as

$$\delta W_E^e = -\int_0^l \rho A\ddot{y}^e\cdot\delta y^e\mathrm{d}\xi - \int_0^l \rho I\ddot{\theta}^e\cdot\delta\theta^e\mathrm{d}\xi + \sum_{s=1}^{n_f} f_s\cdot\delta y^e|_{\xi=\xi_{cs}} \tag{4}$$

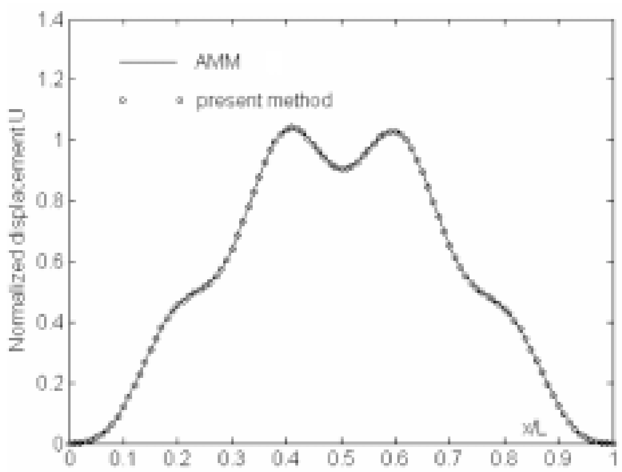

Fig. 3. Normalized displacement under a moving force with $\alpha = 0.11$ and a beam with $\beta = 0.03$.

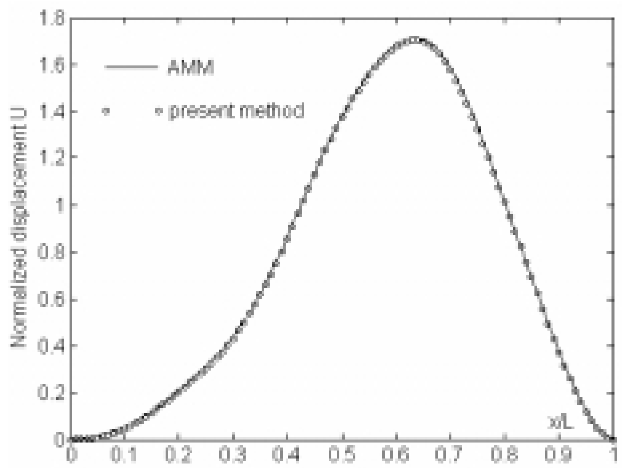

Fig. 4. Normalized displacement under a moving force with α=0.50 and a beam with $\beta = 0.15$.

where ρ is the mass density of the beam material; f_s is the magnitude of the $s-$th concentrated force running on the beam element; ξ_{cs} is the distance between the $s-$th concentrated force and the left node of the beam element; and the dot above symbol denotes the differentiation with respect to time t. Using Eqs (1) and (2), the expressions of the differentiations with respect to local coordinate ξ and time t for y^e and θ^e in Eqs (3) and (4) can be written as

$$\frac{\partial \theta^e}{\partial \xi} = [N'_\theta]\{q\}^e \quad \delta\left(\frac{\partial \theta^e}{\partial \xi}\right) = [N'_\theta]\cdot\delta\{q\}^e \tag{5a,b}$$

$$\frac{\partial y^e}{\partial \xi} - \theta^e = \left[N'_y\right]\{q\}^e - [N_\theta]\{q\}^e \quad \delta\left(\frac{\partial y^e}{\partial \xi} - \theta^e\right) = \left[N'_y\right]\cdot\delta\{q\}^e - [N_\theta]\cdot\delta\{q\}^e \tag{5c,d}$$

$$\ddot{y}^e = [N_y]\{\ddot{q}\}^e \quad \delta y^e = [N_y]\cdot\delta\{q\}^e \tag{6a,b}$$

$$\ddot{\theta}^e = [N_\theta]\{\ddot{q}\}^e \quad \delta\theta^e = [N_\theta]\cdot\delta\{q\}^e \tag{6c,d}$$

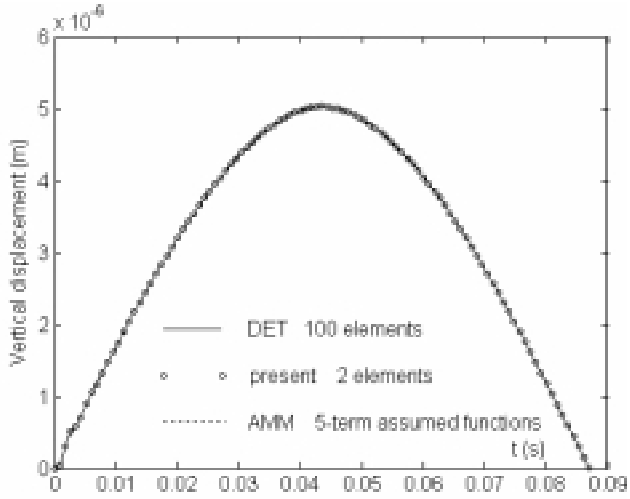

Fig. 5. Time history of dynamic vertical displacement of beam mid-point with three different methods.

2.3. The equation of motion of a Timoshenko beam element under a few moving concentrated forces

Substituting Eqs (5a–d) into Eq. (3) and Eqs (6a–d) into Eq. (4), respectively, then using the principle of virtual work [5], i.e., $\delta W_I^e = \delta W_E^e$, one can obtain the equation of motion for a Timoshenko beam element under a few moving concentrated forces. The equation of motion in matrix form can be written as

$$[\mathrm{M}]^e \{\ddot{q}\}^e + [K]^e \{q\}^e = \sum_{s=1}^{n_f} [N_y]^{\mathrm{T}}_{\xi=\xi_{cs}} \cdot f_s \tag{7}$$

where $[M]^e$ and $[K]^e$ denote the mass and stiffness matrices of the Timoshenko beam element, respectively; $\{\ddot{q}\}^e$ and $\{q\}^e$ denote the nodal acceleration and displacement vectors of the element, respectively; $[N_y]^{\mathrm{T}}_{\xi=\xi_{cs}} \cdot f_s$ denotes the equivalent nodal force vector of the element contributed by the $s-$th concentrated force f_s running on the element; and the superscript T denotes the transpose. $[M]^e$ and $[K]^e$ can be expressed as

$$[M]^e = [M_t]^e + [M_r]^e$$

$$[K]^e = [K_b]^e + [K_s]^e$$

in which

$$[M_t]^e = \int_0^l \rho A [N_y]^T [N_y] \mathrm{d}\xi = \text{ consistent mass matrix for translational inertia}$$

$$[M_r]^e = \int_0^l \rho I [N_\theta]^T [N_\theta] \mathrm{d}\xi = \text{ consistent mass matrix for rotatory inertia}$$

$$[K_b]^e = \int_0^l EI [N'_\theta]^T [N'_\theta] \mathrm{d}\xi = \text{ bending stiffness matrix}$$

$$[K_s]^e = \int_0^l k_s GA ([N'_y]^T - [N_\theta]^T)([N'_y] - [N_\theta]) \mathrm{d}\xi = \text{ shear stiffness matrix}$$

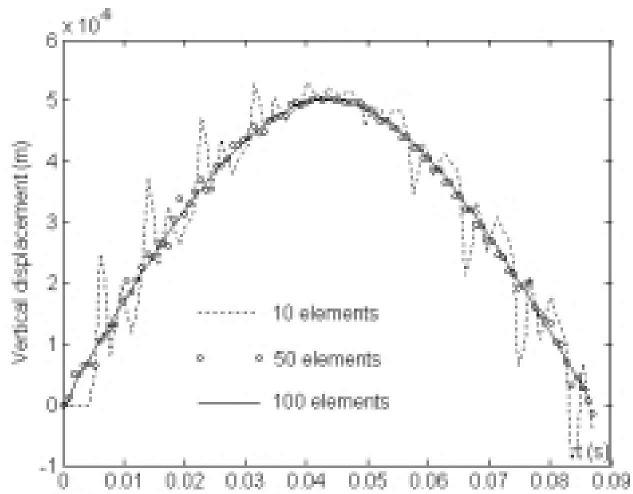

Fig. 6. Time history of dynamic vertical displacement of beam mid-point using DET with various element numbers.

The explicit expressions of the respective element matrices are listed in the Appendix reported by Yokoyama [26]. If $\Phi = 0$ in the element stiffness and mass matrices, and $[M_r]^e$ in Eq. (7) is neglected, Eq. (7) can be degenerated into the equation of motion of an elastic Bernoulli-Euler beam element under a few moving concentrated forces.

2.4. The equation of motion of a Timoshenko beam under a series of moving concentrated forces

By assembling element matrices and element nodal vectors, respectively, one can obtain the global equation of motion for a Timoshenko beam neglecting the damping effect subjected to a series of moving concentrated forces, which will appear as

$$[M]\{\ddot{q}\} + [K]\{q\} = \{F\} \tag{8}$$

where the matrices $[M]$ and $[K]$ are the global mass and stiffness matrices, respectively, of the Timoshenko beam; the vectors $\{\ddot{q}\}$ and $\{q\}$ are the nodal acceleration and displacement vectors, respectively, of the beam; and the vector $\{F\}$ is the equivalent nodal force vector of the beam. Equation (8) can then be solved by the Wilson θ method or similar methods [3]. It should be noted that Eq. (8) could successfully analyze the dynamic response of an elastic Timoshenko or Bernoulli-Euler beam with various boundary conditions, including intermediate supports, subjected to moving concentrated forces.

3. Verification of the present method

For the purpose of verification, let us consider a simply supported Timoshenko beam neglecting the damping effect of the beam subjected to a moving concentrated force. Initially, the beam is at rest, and the moving concentrated force is at the left end of the beam. In the finite element analysis, 6 elements with equal lengths are used for the beam. The equation of motion for the Timoshenko beam subjected to a moving concentrated force is solved by the Wilson θ method with $\theta = 1.4$ and using 100 equal time steps. The same Timoshenko beam parameters, material properties as defined in Lee [13] are used in the numerical simulation. These properties are L (beam length) $=$ 1 m, $E = 2.07 \times 10^{11}$ N/m^2, $G = 7.76 \times 10^{10}$ N/m^2, $k_s = 0.9$, and $\rho = 7700$ kg/m^3. The cross-sectional area of the beam, A, is computed from the radius of gyration r_0 defined by a non-dimensional parameter (Rayleigh's coefficient) $\beta = r_0\pi/L$. The magnitude of the concentrated force is $f_0 = 0.2\rho A L g$. The prescribed axial speed of the moving concentrated force, v, is similarly defined by a non-dimensional parameter given by $\alpha = v/v_{cr}$, where v_{cr} for a supported beam is given by $v_{cr} = (\pi/L)\sqrt{EI/\rho A}$. It should be pointed out that v_{cr} is the speed of a constant force moving on a simply supported Bernoulli-Euler beam when the value of α is equal to 1.

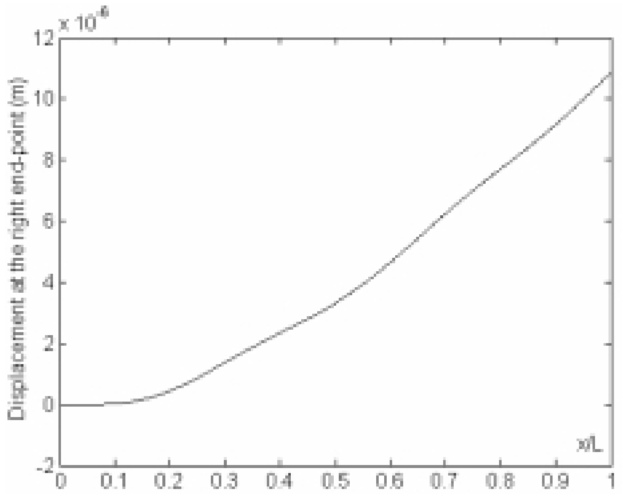

Fig. 7. Displacement at the right end-point of a fixed-free beam with $\beta = 0.15$ under a moving force with $v = 50$ m/s.

This normalized displacement under the moving force is denoted by U, defined by $U = y/y_{st}$. y is the beam displacement in the vertical direction under this moving force evaluated at x, and y_{st} is the beam static displacement at mid-point when the force is applied at the same point for a Bernoulli-Euler beam. From the static analysis, this displacement y_{st} is $f_0L^3/48EI$. The reason for performing this normalization is to facilitate the comparisons of the present result with the corresponding reported work [13]. For the case of Timoshneko beam, the normalized displacement under the moving force with $\alpha = 0.11$ and a beam with $\beta = 0.03$ and the moving force with $\alpha = 0.50$ and a beam with $\beta = 0.15$ have been plotted in Figs 3 and 4, respectively; along with the results obtained by AMM [13] using ten-term assumed functions for the vertical displacement and rotation of the Timoshenko beam. Evidently, good agreements have been achieved between the present results and the reported results [13]. This example serves to illustrate the reliability of the present method.

4. Numerical examples

4.1. Example 1. A simply supported Timoshenko beam subjected to a moving concentrated force

In order to compare the present method with DET presented by Yavari et al. [24], a simply supported Timoshenko beam subjected to a moving concentrated force with constant speed $v = 50$ m/s will be investigated. The parameters of the beam are as follows: $L = 4.352$ m, $E = 2.02 \times 10^{11}$ N/m^2, $G = 7.7 \times 10^{10}$ N/m^2, $A = 0.2025$ m^2, $k_s = 1/1.18$, and m_b (total mass of beam) = 87.04 kg. The magnitude of the concentrated force is 1960 N. These parameters are from reference [24]. The equation of motion for this example is solved by the Wilson θ method with $\theta = 1.4$ and using 100 equal time steps. The time history of dynamic vertical displacement of beam mid-point obtained by the present method with 2 elements having equal lengths has been plotted in Fig. 5, along with the result obtained by DET with 100 elements having equal lengths. The result obtained by AAM using five-term assumed functions for the vertical displacement and rotation of the Timoshenko beam has also been plotted in Fig. 5. As shown in Fig. 5, good agreements have been achieved among three type results. The time histories of dynamic vertical displacement of beam mid-point obtained by DET with 10, 50 and 100 elements have been plotted in Fig. 6. It can be seen from Fig. 6 that the solution obtained by DET converges to an exact answer as the number of elements increases. From Figs 5 and 6, one can draw a conclusion that the present method is more effective than DET. This is because the results obtained by the present method only using 2 elements are agreement with those obtained DET using 100 elements, and the differences between the results obtained by the present method using 2 elements and those obtained by DET using 10 elements are high.

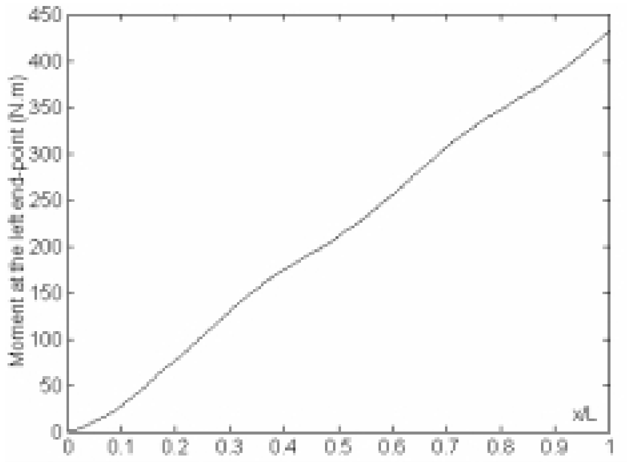

Fig. 8. Moment at the left end-point of a fixed-free beam with $\beta = 0.15$ under a moving force with $v = 50$ m/s.

It should be pointed that reference [24] analyzed the dynamic response of Timoshenko beams under a moving mass using DET. If the formulations presented by reference [24] are used to analyze the dynamic response of Timoshenko beam under a moving concentrated force, $\mathbf{H}$ in Eq. (25) on page 148 and Eq. (32) on page 149 in reference [24] must be deleted. In addition, the shear coefficient of beam in the present method is the reciprocal of the shear factor of beam in reference [24].

4.2. Example 2. A Timoshenko beam with other boundary conditions subjected to a moving concentrated force

The present method can also be applied to analyze the dynamic response of a beam with various boundary conditions, including intermediate supports, subjected to moving concentrated forces. As a numerical example, let us consider a cantilever (fixed-free) Timoshenko beam subjected to a moving concentrated force. It is assumed that the force moves from fixed (left) end to free (right) end of the beam. The parameters in this example are same as those in Section 3 except the speed of the moving force $v = 50$ m/s and Rayleigh's coefficient of the beam $\beta = 0.15$. In the finite element analysis, 6 elements with equal lengths are used for the beam, and the equation of motion is solved by the Wilson θ method with $\theta = 1.4$ and using 100 equal time steps. The displacement at the right end-point of the beam and the moment at the left end-point have been plotted in Figs 7 and 8, respectively.

5. Concluding remarks

Finite element formulations for an elastic Timoshenko beam subjected to moving concentrated forces have been presented in this paper. The correctness of the present method has been illustrated by a comparison with the existing literature. The advantages of the present method are as follows:

(1). The scope of application of the present method is more wide than that of the assumed mode method presented by Lee [13]. The present method can not only analyze the dynamic problem of a simply supported Timoshenko beam subjected to moving concentrated forces, but also analyze the dynamic problem of a Timoshenko beam with various boundary conditions, including intermediate supports.

(2). Numerical results show that the present method is more effective than the discrete element technique presented by Yavari et al. [24] for investigating the dynamic problem of Timoshenko beam.

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Appendix

Element stiffness matrices $[K_b]^e$ and $[K_s]^e$ are as follows

$$[K_b]^e = \frac{EI}{l^3(1+\Phi)^2}\begin{bmatrix} 12 & 6l & -12 & 6l \\ & (4+2\Phi+\Phi^2)l^2 & -6l & (2-2\Phi-\Phi^2)l^2 \\ & & 12 & -6l \\ \text{symm.} & & & (4+2\Phi+\Phi^2)l^2 \end{bmatrix}$$

$$[K_s]^e = \frac{k_s GA\Phi^2}{4l(1+\Phi)^2}\begin{bmatrix} 4 & 2l & -4 & 2l \\ & l^2 & -2l & l^2 \\ & & 4 & -2l \\ \text{symm.} & & & l^2 \end{bmatrix}$$

Element mass matrices $[M_t]^e$ and $[M_r]^e$ are as follows

$$[M_t]^e = \frac{\rho Al}{(1+\Phi)^2}\begin{bmatrix} \frac{13}{35}+\frac{7\Phi}{10}+\frac{\Phi^2}{3} & \left(\frac{11}{210}+\frac{11\Phi}{120}+\frac{\Phi^2}{24}\right)l & \frac{9}{70}+\frac{3\Phi}{10}+\frac{\Phi^2}{6} & -\left(\frac{13}{420}+\frac{3\Phi}{40}+\frac{\Phi^2}{24}\right)l \\ & \left(\frac{1}{105}+\frac{\Phi}{60}+\frac{\Phi^2}{120}\right)l^2 & \left(\frac{13}{420}+\frac{3\Phi}{40}+\frac{\Phi^2}{24}\right)l & -\left(\frac{1}{140}+\frac{\Phi}{60}+\frac{\Phi^2}{120}\right)l^2 \\ & & \frac{13}{35}+\frac{7\Phi}{10}+\frac{\Phi^2}{3} & -\left(\frac{11}{210}+\frac{11\Phi}{120}+\frac{\Phi^2}{24}\right)l \\ \text{symm.} & & & \left(\frac{1}{105}+\frac{\Phi}{60}+\frac{\Phi^2}{120}\right)l^2 \end{bmatrix}$$

$$[M_r]^e = \frac{\rho Al}{(1+\Phi)^2}\left(\frac{r_0}{l}\right)^2\begin{bmatrix} \frac{6}{5} & \left(\frac{1}{10}-\frac{\Phi}{2}\right)l & -\frac{6}{5} & \left(\frac{1}{10}-\frac{\Phi}{2}\right)l \\ & \left(\frac{2}{15}+\frac{\Phi}{6}+\frac{\Phi^2}{3}\right)l^2 & -\left(\frac{1}{10}-\frac{\Phi}{2}\right)l & -\left(\frac{1}{30}+\frac{\Phi}{6}-\frac{\Phi^2}{6}\right)l^2 \\ & & \frac{6}{5} & -\left(\frac{1}{10}-\frac{\Phi}{2}\right)l \\ \text{symm.} & & & \left(\frac{2}{15}+\frac{\Phi}{6}+\frac{\Phi^2}{3}\right)l^2 \end{bmatrix}$$

in which $r_0 = \sqrt{I/A}$ = radius of gyration.

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